MATHEMATICS
BOOK 2

HAPPINESS

IS

MATHEMATICS

MATHEMATICS

BOOK TWO

INCLUDE UNDER
SEQUENCES

SEQUENCES OF
PYRAMIDS

MATHEMATICS
BOOK TWO

NUMBER SEQUENCES AND SERIES
RECURSIVE AND EXPLICIT FORMULAE
Pythagorean ~~Triple~~ Triples

~~NUMBER ARRAYS~~

~~SPECIAL~~ NUMBER~~S~~ SEQUENCES
FIBONACCI ~~SEQUENCES AND ARRAYS~~
RAMANUJAN ~~SEQUENCES AND ARRAYS~~
FULCRUM ~~SEQUENCES AND ARRAYS~~
OTHER SEQUENCES ~~AND ARRAYS~~
~~BELL~~ SQUARES etc
Triangular ...

~~YANGHUIS~~ PRIMES
Pythagorean Triple
SERIES
SUMMATIONS
GAUSS INVERSION METHOD
POWER SERIES

CONTINUED FRACTIONS
CONTINUED ROOT

PRODSUM NUMBERS

ONE-DIMENSIONAL
SEQUENCES LINEAR PATTERNS

WHAT RULE DETERMINES THE NEXT MEMBER OF THE SEQUENCE

WHAT RULE DETERMINES THE N^{th} MEMBER OF THE SEQUENCE

WHAT RULE CONNECTS MEMBERS OF THE SEQUENCE

- ; PURE SEQUENCE
- + SERIES
- × PRODUCT
- ÷ CONTINUOUS FRACTION
- $\sqrt{\ }$
- MIX ...

SEQUENCES

RECURSION $A_{n+2} = b A_{n+1} + c A_n$ — 2 input

ITERATION $A_{n+1} = \sqrt{bA_n + c}$, $A_{n+1} = \frac{A_n^2 - c}{b}$ — 1 input

EXPLICIT $\frac{v^n - u^n}{v - u}$ — n pos

SUMMATION n

CROSS-SEQUENCES e.g. B, R, K

$B_{2n} = R_{n+1} + K_n$ tell the future

Special Cases

Fulcrum Numbers

Ramanujan Sequency

$\Pi \Sigma$ e.g. 10

$\Pi \Delta$ e.g Fibonacci

Bell Numbers

Arrangements

PYTHAGOREAN TRIPLES

$m := 1, 2 .. 6$ $\quad$ $n := 1, 2 .. 10$

$A_{m,n} := 2 \cdot m \cdot n$ $\quad$ $B_{m,n} := m^2 - n^2$ $\quad$ $C_{m,n} := m^2 + n^2$

m

B =

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-3	-8	-15	-24	-35	-48	-63	-80	-99
2	0	3	0	-5	-12	-21	-32	-45	-60	-77	-96
3	0	8	5	0	-7	-16	-27	-40	-55	-72	-91
4	0	15	12	7	0	-9	-20	-33	-48	-65	-84
5	0	24	21	16	9	0	-11	-24	-39	-56	-75
6	0	35	32	27	20	11	0	-13	-28	-45	-64

$B^2 + A^2 = C^2$

A =

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	4	6	8	10	12	14	16	18	20
2	0	4	8	12	16	20	24	28	32	36	40
3	0	6	12	18	24	30	36	42	48	54	60
4	0	8	16	24	32	40	48	56	64	72	80
5	0	10	20	30	40	50	60	70	80	90	100
6	0	12	24	36	48	60	72	84	96	108	120

C =

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	5	10	17	26	37	50	65	82	101
2	0	5	8	13	20	29	40	53	68	85	104
3	0	10	13	18	25	34	45	58	73	90	109
4	0	17	20	25	32	41	52	65	80	97	116
5	0	26	29	34	41	50	61	74	89	106	125
6	0	37	40	45	52	61	72	85	100	117	136

↑m m↓

SEQUENCES

RECURSION FORMULAE   INITIAL VALUES

CHARACTERISTIC EQUATIONS

EXPLICIT FORMULAE

RESONANCES, MUTATIONS

DIFFERENCE SEQUENCES

SUMMATION SEQUENCES

RECURSION-
EXPLICT

General Recursion Formula:

$$U_{n+2} = J U_{n+1} + K U_n$$

$x^2 - Jx - K = 0$

$p = \frac{J + \sqrt{J^2+4K}}{2}$, $q = \frac{J - \sqrt{J^2+4K}}{2}$, $p + q = J$

FROBSUM: $K = -J$ $\qquad$ $p - q = \sqrt{J^2+4K}$ $\qquad$ $p \times q = -K$

ΠΔ $K = J$ $\qquad$ $p^2 + q^2 = J^2 + 8K$

Initial Values

$U_1 = a$ $\qquad$ $U_5 = (J^3 + 2JK)b + (J^2K + K)a$

$U_2 = b$ $\qquad$ $U_6 = (J^4 + 3J^2K + K^2)b + (J^3K + JK + JK^2)a$

$U_3 = Jb + Ka$

$U_4 = J(Jb + Ka) + Kb = (J^2 + K)b + JKa$

Explicit: $U_n = c p^n + d q^n$

$U_1 = a = cp + dq$ $\qquad$ $aq = cpq + dq^2$

$U_2 = b = cp^2 + dq^2$ $\qquad$ $aq - b = cp(q-p)$; $\quad c = \frac{aq-b}{p(q-p)}$

$d = \frac{a - cp}{q} = \frac{ap-b}{q(p-q)}$

$$U_n = \frac{b-aq}{p(p-q)} p^n + \frac{ap-b}{q(p-q)} q^n$$

Explicit Formula:

$$U_n = \frac{1}{p-q}\left[(b-aq)p^{n-1} + (ap-b)q^{n-1}\right]$$

$$= \frac{1}{p-q}\left[b(p^{n-1} - q^{n-1}) - a(qp^{n-1} - pq^{n-1})\right]$$

EXAMPLE: $J = K = 1$, $p = \frac{1+\sqrt{5}}{2}$, $q = \frac{1-\sqrt{5}}{2}$

$a = 1$, $b = 1$ $\quad$ $p + q = 1$, $1 - q = p$, $1 - p = q$ $\quad$ $U_n = \frac{p^n - q^n}{p-q} = \frac{p^n - q^n}{\sqrt{5}}$ $\quad n = 1, 2,$

$p = \Phi$, $q = \varphi$

$a = 0$, $b = 1$ $\quad$ $U_n = \frac{p^{n-1} - q^{n-1}}{p-q}$ $\quad n = 1, 2, \ldots$

EXAMPLE $J = K = 1$

$a = 1$, $b = 3$ $\quad p + q = 1$ $\quad p$

$$U_n = \frac{1}{p-q}\left[(3-q)p^{n-1} + (p-3)q^{n-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[(p+2)p^{n-1} - (2+q)q^{n-1}\right]$$

$$p+2 = \frac{5+\sqrt{5}}{2} \qquad \frac{p+2}{\sqrt{5}} = \frac{1+\sqrt{5}}{2} = p$$

$$-(q+2) = \frac{\sqrt{5}-5}{2} \qquad \frac{-(2+q)}{\sqrt{5}} = \frac{1-\sqrt{5}}{2} = q$$

$$\therefore U_n = p^n + q^n$$

RECURSION FORMULA

CHARACTERISTIC EQUATION

EXPLICIT FORMULA

RECEXP10.MCD 06-08-09

RECURSION FORMULAE and EXPLICIT FORMULAE

Sequences when Roots are Unequal

FIBONACCI SEQUENCE

$u := 1$ $v := 1$

$n := 0, 1 .. 15$ $j := 1$ $k := 1$ $A_0 := 1$ $A_1 := 1$

RECURSION $A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$

EXPLICIT Case of unequal roots with initial values a, b $a := u$ $b := v$

$r := \sqrt{j^2 + 4 \cdot k}$ $r = 2.236068$ $r^2 = 5$

$p := \frac{(j+r)}{2}$ $q := \frac{(j-r)}{2}$

$p = 1.618034$ $q = -0.618034$

$$E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p-q)} \qquad G_n := \frac{\left[(b - a \cdot q) \cdot p^{n-1} - (b - a \cdot p) \cdot q^{n-1}\right]}{(p-q)}$$

$n =$	$A_n =$	$E_n =$	$G_n =$
0	1	1	0
1	1	1	1
2	2	2	1
3	3	3	2
4	5	5	3
5	8	8	5
6	13	13	8
7	21	21	13
8	34	34	21
9	55	55	34
10	89	89	55
11	144	144	89
12	233	233	144
13	377	377	233
14	610	610	377
15	987	987	610

RECURSION FORMULAE and EXPLICIT FORMULAE

Sequences when Roots are Unequal

LUCAS SEQUENCE

$u := 1 \qquad v := 3$

$n := 0, 1 .. 15 \qquad j := 1 \qquad k := 1 \qquad A_0 := 1 \qquad A_1 := 3$

RECURSION $\qquad A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$

EXPLICIT $\qquad$ Case of unequal roots with initial values a, b $\qquad a := u \qquad b := v$

$r := \sqrt{j^2 + 4 \cdot k} \qquad r = 2.236068 \qquad r^2 = 5$

$p := \frac{(j + r)}{2} \qquad q := \frac{(j - r)}{2}$

$p = 1.618034 \qquad q = -0.618034$

$$E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p - q)} \qquad G_n := \frac{\left[(b - a \cdot q) \cdot p^{n-1} - (b - a \cdot p) \cdot q^{n-1}\right]}{(p - q)}$$

$n =$	$A_n =$	$E_n =$	$G_n =$
0	1	1	2
1	3	3	1
2	4	4	3
3	7	7	4
4	11	11	7
5	18	18	11
6	29	29	18
7	47	47	29
8	76	76	47
9	123	123	76
10	199	199	123
11	322	322	199
12	521	521	322
13	843	843	521
14	1364	1364	843
15	2207	2207	1364

RECURSION FORMULAE and EXPLICIT FORMULAE

Sequences when Roots are Unequal

RAMANUJAN SEQUENCE

$u := 0 \qquad v := 1$

$n := 0, 1 .. 15 \qquad j := 6 \qquad k := -1 \qquad A_0 := 0 \qquad A_1 := 1$

RECURSION $\qquad A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$

EXPLICIT $\qquad$ Case of unequal roots with initial values a, b $\qquad a := u \qquad b := v$

$r := \sqrt{j^2 + 4 \cdot k} \qquad r = 5.656854 \qquad r^2 = 32$

$p := \frac{(j + r)}{2} \qquad q := \frac{(j - r)}{2}$

$p = 5.828427 \qquad q = 0.171573$

$$E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p - q)} \qquad G_n := \frac{\left[(b - a \cdot q) \cdot p^{n-1} - (b - a \cdot p) \cdot q^{n-1}\right]}{(p - q)}$$

n =	A_n =	E_n =	G_n =
0	0	0	-1
1	1	1	0
2	6	6	1
3	35	35	6
4	204	204	35
5	1189	1189	204
6	6930	6930	1189
7	40391	40391	6930
8	$2.35416 \cdot 10^5$	$2.35416 \cdot 10^5$	40391
9	$1.372105 \cdot 10^6$	$1.372105 \cdot 10^6$	$2.35416 \cdot 10^5$
10	$7.997214 \cdot 10^6$	$7.997214 \cdot 10^6$	$1.372105 \cdot 10^6$
11	$4.661118 \cdot 10^7$	$4.661118 \cdot 10^7$	$7.997214 \cdot 10^6$
12	$2.716699 \cdot 10^8$	$2.716699 \cdot 10^8$	$4.661118 \cdot 10^7$
13	$1.583408 \cdot 10^9$	$1.583408 \cdot 10^9$	$2.716699 \cdot 10^8$
14	$9.228778 \cdot 10^9$	$9.228778 \cdot 10^9$	$1.583408 \cdot 10^9$
15	$5.378926 \cdot 10^{10}$	$5.378926 \cdot 10^{10}$	$9.228778 \cdot 10^9$

RECURSION FORMULAE and EXPLICIT FORMULAE

Sequences when Roots are Unequal

PHYSICAL CONSTANTS SEQUENCE

$u := 0 \qquad v := 1$

$n := 0, 1 .. 15 \qquad j := 10 \qquad k := -10 \qquad A_0 := 0 \qquad A_1 := 1$

RECURSION $\qquad A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$

EXPLICIT $\qquad$ Case of unequal roots with initial values a, b $\qquad a := u \qquad b := v$

$r := \sqrt{j^2 + 4 \cdot k} \qquad r = 7.745967 \qquad r^2 = 60$

$p := \frac{(j + r)}{2} \qquad q := \frac{(j - r)}{2}$

$p = 8.872983 \qquad q = 1.127017 \qquad \frac{p^2}{2} = 39.364917$

$E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p - q)} \qquad G_n := \frac{\left[(b - a \cdot q) \cdot p^{n-1} - (b - a \cdot p) \cdot q^{n-1}\right]}{(p - q)}$

n =	A_n =	E_n =	G_n =
0	0	0	-0.1
1	1	1	0
2	10	10	1
3	90	90	10
4	800	800	90
5	7100	7100	800
6	63000	63000	7100
7	$5.59 \cdot 10^5$	$5.59 \cdot 10^5$	63000
8	$4.96 \cdot 10^6$	$4.96 \cdot 10^6$	$5.59 \cdot 10^5$
9	$4.401 \cdot 10^7$	$4.401 \cdot 10^7$	$4.96 \cdot 10^6$
10	$3.905 \cdot 10^8$	$3.905 \cdot 10^8$	$4.401 \cdot 10^7$
11	$3.4649 \cdot 10^9$	$3.4649 \cdot 10^9$	$3.905 \cdot 10^8$
12	$3.0744 \cdot 10^{10}$	$3.0744 \cdot 10^{10}$	$3.4649 \cdot 10^9$
13	$2.72791 \cdot 10^{11}$	$2.72791 \cdot 10^{11}$	$3.0744 \cdot 10^{10}$
14	$2.42047 \cdot 10^{12}$	$2.42047 \cdot 10^{12}$	$2.72791 \cdot 10^{11}$
15	$2.147679 \cdot 10^{13}$	$2.147679 \cdot 10^{13}$	$2.42047 \cdot 10^{12}$

RECURSION FORMULAE

2 values determining a third

General Case $A_{n+2} = a A_{n+1} + b A_n$

$\rightarrow \quad x^2 - ax - b = 0$

roots $\quad x_1 = \frac{a + \sqrt{a^2+4b}}{2}$, $x_2 = \frac{a - \sqrt{a^2+4b}}{2}$

Fibonacci Case $A_{n+2} = A_{n+1} + A_n$ 0, 1, 1, 2, 3, 5, 8, 13, ...

$a = 1$ $\quad x^2 - x - 1 = 0$

$b = 1$ $\quad x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2} = -\varphi$ or rather $\frac{\sqrt{5}-1}{2} = \varphi = 0.618034\ldots$

$\Phi = 1.618034$

FULCRUM NUMBERS

RAMANUJAN NUMBERS 0, 1, 6, 35, 204, ...

$B_{n+2} = 6 B_{n+1} - B_n$

$a = 6$ $\quad x^2 - 6x + 1 = 0$

$b = -1$ $\quad x_1 = 3 + \sqrt{8}$, $x_2 = 3 - \sqrt{8}$

PRODSUM NUMBERS

$A_{n+2} = p A_{n+1} - p A_n$ $\quad x_1 + x_2 = p$ $\quad x_1, x_2$ imaginary

$a = p$ $\quad x^2 - px + p = 0$ $\quad x_1 \cdot x_2 = p$ $\quad$ for $p = 1, 2, 3$

$b = -p$

For $p = 1$, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0

$p = 2$, 0, 1, 2, 2, 0, -4, -8, -8, 0, 16, 32, 32, 0 -64, -128

3 0, 1, 3, 6, 9, 9, 0, -27,

4 0, 1, 4, 12, 32, 80, 192

5 0, 1, 5, 20, 75

BODE NUMBERS

$p = 10$ 0, 1, 10, 90, 800, 7100, ...

1, 2, 10, 80, 700 6200, ...

or $$A_{n+2} = L[a A_{n+1} + b A_n]$$

3 fixed numbers L, a, b

In recursion, there are 2 changing numbers
in iteration, 1 changing number

EXPLICIT FORMULAE FROM Δ's ≡ k

$V = \frac{1}{2}[2n^3 - n^2 - n]$

not a recursion formula

It is an explicit formula

n	V
0	0
1	5
2	21
3	54

This example is for squarefilum

V = Σ L = Σ v

e.g. 2 3 (4) 5 → 5

6 7 8 (9) 10 11 → 21

$V = an^3 + bn^2 + cn + d$

$n=0 \Rightarrow d = 0$

$n=1 \quad a + b + c = 0$

$n=2 \quad 8a + 4b + 2c = 5$

$n=3 \quad 27a + 9b + 3c = 21$

$c = -a - b$

$2c = -2a - 2b$

$3c = -3a - 3b$

$6a + 2b = 5$

$24a + 6b = 21$

$-18a - 6b = -5$

$6a = 16$

$a = \frac{8}{3}$

$b =$

$24a + 8b = 20$

$24a + 6b = 21$

$2b = -1$

$b = -\frac{1}{2}$

$6a - 1 = 5$

$a = 1$

$c = -1 + \frac{1}{2} = \frac{1}{2}$

$V = n^3 - \frac{n^2}{2} + \frac{n}{2}$ ✓

V	1°	2°	3°
0			
	5		
5		11	
	16		6
21		17	
	33		6
54		23	
	56		6
110		29	
	85		6
195		35	
	120		
315			

⇒ cubic equation

> The number of differences to reach a constant difference = the degree of the equation

Subj:	**no visit**
Date:	9/21/2005 8:31:33 AM Pacific Daylight Time
From:	DLongenecker9744
To:	AIW1871

I had a change in plans which required me to drive through on #5. I will be back in October for a brief visitl
Dr. Shima says that what I experienced was a 'crashing' of the Yin field which surrounds the kidneys, Yin being in part water. So the task is to re-build the Yin. I ended up with total Yang which is a bit out of balance. Yang is restless and Yin puts dcown roots. I lost my roots.
The visit south was mixed. One old lover is dying of cancer and is suffering intense painl The other greated me warmly but may or may not leave to anything in the futurel We are different people. don

FORMULAE: RECURSIVE TO EXPLICIT[1]

Recursive formula: **1)** $\mathbf{A_n = j\,A_{n-1} + k\,A_{n-2}}$

To derive a solution, assume $A_n = r^n$

$r^n = j\,r^{n-1} + k\,r^{n-2}$

$r^2 = j\,r + k$

Characteristic polynomial: **2)** $\mathbf{r^2 - j\,r - k = 0}$

whose roots are: $p = (j + (j^2 + 4k)^{1/2})/2$, and $q = (j - (j^2 + 4k)^{1/2})/2$

Hence, $A_n = p^n$ and $A_n = q^n$ are solutions of 1)

that is, $p^n = j\,p^{n-1} + k\,p^{n-2}$ and $q^n = j\,q^{n-1} + k\,q^{n-2}$

But a constant times a solution of **1)** is also a solution of **1)**. i.e. if c and d are constants,

then $A_n = c\,p^n$ and $A_n = d\,q^n$ are solutions.

Also the sum of two solutions is a solution, therefore, $A_n = p^n + q^n$ is a solution.

Combining the above, noting that $p = p(j,k)$ and $q = q(j,k)$, we get,

Explicit formula **3)** $\mathbf{A_n = c\,p^n + d\,q^n}$

as the form of the most general formula for A_n .

However, to determine c and d the <u>initial values</u> of **1)** must be known. For example:

If $A_0 = 0$, $n = 0$, then $c + d = 0$ and if $A_1 = 1, n = 1$, then $c(p-q) = 1$

giving, **4)** $\mathbf{A_n = (p^n - q^n)/ (p - q)}$,

Which is the most general explicit equation for $A_0 = 0$ and $A_1 = 1$

[note: $p - q = (j^2 + 4\,k)^{1/2}$]

---

[1] [see Anderson's DISCRETE MATHEMATICS, p 220ff]

FORMULAE: RECURSIVE TO EXPLICIT[1]

Recursive formula: 1) $\mathbf{A_n = j\, A_{n-1} + k\, A_{n-2}}$

assume $A_n = r^n$

$r^n = j\, r^{n-1} + k\, r^{n-2}$

$r^2 = j\, r + k$

Characteristic polynomial: $\mathbf{r^2 - j\, r - k = 0}$

Roots: $p = (j + \sqrt{(j^2 + 4k)})/2$, $q = (j - \sqrt{(j^2 + 4k)})/2$

Hence, $A_n = p^n$ and $A_n = q^n$ are solutions of 1)

that is, $p^n = j\, p^{n-1} + k\, p^{n-2}$ and $q^n = j\, q^{n-1} + k\, q^{n-2}$

A constant times a solution of 1) is also a solution of 1). i.e. if c and d are constants.

then $A_n = c\, p^n = c\, j\, p^{n-1} + c\, k\, p^{n-2}$ and $A_n = d\, q^n = d\, j\, q^{n-1} + d\, k\, q^{n-2}$

Also the sum of two solutions is a solution, therefore, $A_n = p^n + q^n$ and

Explicit equation: 2) $\mathbf{A_n = c\, p^n + d\, q^n}$ is the most general solution.

Giving: $A_0 = c + d$ and $A_1 = c\, p + d\, q$

To determine c and d initial values of 1) must be known, for example:

If $A_0 = 0$, $c + d = 0$ and if $A_1 = 1$, then $c(p-q) = 1$

Hence, 3) $A_n = (p^n - q^n)/ (p - q)$ is the most general explicit equation for

$A_0 = 0$ and $A_1 = 1$ [The values of p and q depend on j and k]

[1] [see DISCRETE MATHEMATICS, p 220ff]

THE CASE OF EQUAL ROOTS

When the roots of the characteristic equation

$$\textbf{2)} \qquad \mathbf{r^2 - j\,r\; - k = 0}$$

of the recursive equation **1)** are equal, then **2)** may be written in the form

$$(r - t)^2 = 0 \qquad \text{or} \quad r^2 - 2\,r\,t + t^2 = 0$$

Hence, $j = 2\,t$ and $k = -\,t^2$, i.e. the solution $t = j/2$. Equation **1)** then becomes

$$A_n = 2\,t\,A_{n-1} - t^2\,A_{n-2}$$

We know that $A_n = t^n$ is a solution of this recursive equation, but $A_n = n\,t^n$ is also a solution, for

$$n\,t^n = 2\,t\,(n-1)\,t^{n-1} - t^2\,(n-2)\,t^{n-2} = t^n\,[\,2(n-1) - (-2)]$$

$$n\,t^n = n\,t^n$$

Thus in the case of equal roots, the general explicit formula becomes

$$\textbf{5)} \qquad \mathbf{A_n = c\,t^n + d\,n\,t^n}$$

Again for the <u>initial values</u> $A_0 = 0$ and $A_1 = 1$, we have $c = 0$ and $d = 1/t$

Hence $\qquad \textbf{6)} \qquad \mathbf{A_n = n\,t^{n-1}}$ $\qquad$ is the general solution

---

For general <u>initial values</u>, $\qquad A_0 = \alpha$ and $A_1 = \beta$,

In the case of unequal roots, **3)** becomes:

$$\textbf{7)} \qquad \mathbf{A_n = [(\beta - \alpha\,q)\,p^n - (\beta - \alpha\,p)\,q^n\,] / (p - q)}$$

In the case of equal roots, **~~6~~ 5)** becomes:

$$\textbf{8)} \qquad \mathbf{A_n = \alpha\,(1\text{-}n)\,t^n + \beta\,n\,t^{n-1}}$$

For $A_0 = 0,\ A_1 = 1$

$$E_n = \frac{p^n - q^n}{p - q} = p^{n-1} q + p^{n-2} q^2 + \cdots + p q^{n-2} + q^{n-1}$$

n terms

If $p = q$ equal roots $E_n = \sum_1^n p^{n-1} = n p^{n-1}$

THIS MATERIAL BELONGS IN BOOK II ✓

PART I 1998 #40
PART II 2004 #39

NUMAPRX4.WPD

September 2, 2004

SOME NUMERICAL APPROXIMATIONS IV

RECURSION FORMULA: $A_{n+2} = 10\ A_{n+1} - 10\ A_n$

CHARACTERISTIC POLYNOMIAL: $r^2 - 10\ r + 10 = 0$

The two roots are: $u = 5 - \sqrt{15} = 1.1270166...$ $u^2 = 1.270166...$
$v = 5 + \sqrt{15} = 8.8729833...$ $v^2 = 78.729833$

$v^2/2 = 39.364917$

$\alpha\mu S = 40.482956$
$25 + 4\sqrt{15} = 40.491934$
$\delta = 0.008978$
$Q = 1.000222$

MEASURED VALUES: $\log_{10}(\alpha\mu) = 1.127074$, $\log_{10} S = 39.355882$

$[\log(\alpha\mu)]^2 = 1.2702958$

$\log_{10}(\alpha\mu) - u = 0.000057$ $Q = 1.000051$

$[\log(\alpha\mu)]^2 - u^2 = 0.000130$
$Q = 1.000102$

$v^2/2 - \log_{10} S = 0.009035$ $Q = 1.000230$

EXPLICIT FORMULA: (see RECEXP9.MCD DESACH)

$A_n = (p^n - q^n)/(p-q)$ ⇒ 0, 1, 10, ...

where $p = 5+\sqrt{15}$, $q = 5 - \sqrt{15}$ and $p - q = \sqrt{60}$

$v^2/10 = 7.872983$ $(\log_{10} S)/5 = 7.871176$

$\delta = 0.001807$ $Q = 1.000230$

$\log(\alpha\mu S^{1/5}) = 9$

$u + v^2/10 = 9$ $\log_{10}(\alpha\mu) + (\log_{10} S)/5 = 8.998250$

$\delta = 0.001750$ $Q = 1.000194$

$v - u = \sqrt{60}$	$v + u = 10$	$v\ u = 10$
$v^2 - u^2 = 10\sqrt{60}$ $(v-u)^2 = 60$	$v^2 + u^2 = 80$ $(v+u)^2 = 100$	$v^2 u^2 = 100$
$v^3 - u^3 = 90\sqrt{60}$	$v^3 + u^3 = 700$	$v^3 u^3 = 1000$

sequence 1, 10, 90, 800, 7100,....x$\sqrt{60}$ sequence 2, 10, 80, 700, 6200, ...

$A_{n+2} = 10\ A_{n+1} - 10\ A_n$ [0,1] $A_{n+2} = 10\ A_{n+1} - 10\ A_n$ ~~[2, 10]~~ [1, 2] {difference x 10} ?

cf Fibonacci vs Lucas sequences

Another number close to the $N^2 = 10(N-1)$ group

besides $5-\sqrt{15}$ and $\log(\alpha\mu)$

is $\zeta = 10^5 \cdot \mu^{-3/2}$ where $\mu = \frac{m_p}{m_e} = 1836.1535$

$\zeta = 1.270974$ $\quad u^2 = 1.270017$ $\quad (\alpha\mu)^2 = 1.1270296$ (logs)

$\zeta^{1/2} = 1.127375$ $\quad u = 1.127017$ $\quad \alpha\mu = 1.127074$

Comparisons with Fibonacci Numbers

$p = 5+\sqrt{15}$ $\qquad \Phi = \frac{1+\sqrt{5}}{2}$

$q = 5-\sqrt{15}$ $\qquad \varphi = \frac{1-\sqrt{5}}{2}$

EXPLICIT $A_n = \frac{p^n - q^n}{\sqrt{60}}$ $\qquad F_n = \frac{\Phi^n - \varphi^n}{\sqrt{5}}$

RECURSIVE $A_{n+1} = 10A_n - 10A_{n-1}$ $\qquad F_{n+1} = F_n + F_{n-1}$

102 870

12 90 780 6900

M=10 [1,2]: 2 10 80 700 6200

8 70 620 5500

62 550 4980

488

SEE RHOMBOIDS IN BOOK III

M=10 [0,1]: 1 10 90 800 7100

9 80 710 6300

CONSIDER THE ~~SET~~ CLASS $A_{n+2} = MA_{n+1} \pm MA_n$

Class $A_{n+2} = MA_{n+1} \pm KA_n$

RAMANUJAN

$A_{n+2} = 6A_{n+1} - A_n$

Init [0,1] $\quad M=1, +$: Fibonacci $A_{n+2} = A_{n+1} + A_n$

$M=10, -$: $\sim \alpha\mu, S$

all ok for (+), $M = 1, 2, \ldots$

For (−) $M=1, \rightarrow$ oscilate

$M = 2, 3$ "

$M =$ ~~4~~ 4 takes off 1, 4, 12, 32, 80, 192, ...

RECURSION FORMULAE and EXPLICIT FORMULAE

General Unequal Roots Sequences

$j := 1$ $k := 1$ $A_0 := 0$ $A_1 := 1$ $n := 0, 1 .. 15$

RECURSION $A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$ FIBONACCI

EXPLICIT Case of unequal roots and initial values a, b $a := 0$ $b := 1$

$r := \sqrt{j^2 + 4 \cdot k}$ $r = 2.23606798$

$p := \frac{(j + r)}{2}$ $q := \frac{(j - r)}{2}$

$p = 1.61803399$ $q = -0.61803399$

$$E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p - q)}$$

$n =$	$E_n =$	$A_n =$
0	0	0
1	1	1
2	1	1
3	2	2
4	3	3
5	5	5
6	8	8
7	13	13
8	21	21
9	34	34
10	55	55
11	89	89
12	144	144
13	233	233
14	377	377
15	610	610

RECURSION FORMULAE and EXPLICIT FORMULAE

Sequences when Roots are Unequal

$n := 0, 1 .. 15 \quad j := 1 \quad k := 1 \quad A_0 := 0 \quad A_1 := 1 \quad a := 0 \quad b := 1$

$r := \sqrt{j^2 + 4 \cdot k} \qquad p := \frac{(j + r)}{2} \qquad q := \frac{(j - r)}{2}$

RECURSION $A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$ EXPLICIT $E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p - q)}$

n =	A_n =	E_n =
0	0	0
1	1	1
2	1	1
3	2	2
4	3	3
5	5	5
6	8	8
7	13	13
8	21	21
9	34	34
10	55	55
11	89	89
12	144	144
13	233	233
14	377	377
15	610	610

RECURSION FORMULAE and EXPLICIT FORMULAE

General Unequal Roots Sequences

$j := 6$ $k := -1$ $A_0 := 0$ $A_1 := 1$ $n := 0, 1 .. 15$

RECURSION $A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$ RAMANUJAN

EXPLICIT Case of unequal roots and initial values a, b $a := 0$ $b := 1$

$r := \sqrt{j^2 + 4 \cdot k}$ $\sqrt{32}$ $r = 5.65685425$

$p := \frac{(j + r)}{2}$ $q := \frac{(j - r)}{2}$

$p = 5.82842712$ $q = 0.17157288$

$$E_n := \frac{\left[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n\right]}{(p - q)}$$

$n =$	$E_n =$	$A_n =$
0	0	0
1	1	1
2	6	6
3	35	35
4	204	204
5	1189	1189
6	6930	6930
7	40391	40391
8	235416	235416
9	1372105	1372105
10	7997214	7997214
11	$4.6611179 \cdot 10^7$	$4.6611179 \cdot 10^7$
12	$2.7166986 \cdot 10^8$	$2.7166986 \cdot 10^8$
13	$1.58340798 \cdot 10^9$	$1.58340798 \cdot 10^9$
14	$9.22877803 \cdot 10^9$	$9.22877803 \cdot 10^9$
15	$5.37892602 \cdot 10^{10}$	$5.37892602 \cdot 10^{10}$

RECURSION FORMULAE and EXPLICIT FORMULAE

Sequences when Roots
are Unequal

$n := 0, 1 .. 15$ $j := 10$ $k := -10$ $A_0 := 2$ $A_1 := 10$

RECURSION $A_{n+2} := j \cdot A_{n+1} + k \cdot A_n$ DESACH

EXPLICIT Case of unequal roots with initial values a, b $a := 0$ $b := 1$

$r := \sqrt{j^2 + 4 \cdot k}$ $r = 7.74596669$

$p := \frac{(j+r)}{2}$ $q := \frac{(j-r)}{2}$

$p = 8.87298335$ $q = 1.12701665$

$p + q = j$
$r + q = p$
$r = \sqrt{60}$
$p = 5 + \sqrt{15}$
$q = 5 - \sqrt{15}$

$m := 1, 2 .. 15$

$E_n := \frac{[(b - a \cdot q) \cdot p^n - (b - a \cdot p) \cdot q^n]}{(p - q)}$ $L_m := \log(A_m)$

n =	E_n =	A_n =	L_m =
0	0	2	1
1	1	10	1.90308999
2	10	80	2.84509804
3	90	700	3.79239169
4	800	6200	4.74036269
5	7100	55000	5.68841982
6	63000	488000	6.6364879
7	559000	4330000	7.58455736
8	4960000	$3.842 \cdot 10^{7}$	8.532627
9	$4.401 \cdot 10^{7}$	$3.409 \cdot 10^{8}$	9.48069666
10	$3.905 \cdot 10^{8}$	$3.0248 \cdot 10^{9}$	10.42876633
11	$3.4649 \cdot 10^{9}$	$2.6839 \cdot 10^{10}$	11.376836
12	$3.0744 \cdot 10^{10}$	$2.38142 \cdot 10^{11}$	12.32490566
13	$2.72791 \cdot 10^{11}$	$2.11303 \cdot 10^{12}$	13.27297533
14	$2.42047 \cdot 10^{12}$	$1.874888 \cdot 10^{13}$	14.221045
15	$2.147679 \cdot 10^{13}$	$1.663585 \cdot 10^{14}$	

$G := L_{15} - L_{14}$ $G = 0.94806967$ $10^G = 8.87298335$ $10 \cdot 10^{-G} = 1.12701665$

RECEXP1.MCD 04-02-17

RECURSION FORMULAE TO EXPLICIT FORMULAE

A(n+1) = j A(n) + k A(n-1)

Case of unequal roots with initial values A(0)=0, A(1)=1

$j := 0, 1 .. 9$ $k := 0$ $n := 0, 1 .. 18$ NUMBER SEQUENCE

$r(j) := \sqrt{j^2 + 4 \cdot k}$

$p(j) := \frac{(j + r(j))}{2}$ $q(j) := \frac{(j - r(j))}{2}$

a b j k
0 1 j 0

$A_{n,j} := \frac{\left(p(j)^n - q(j)^n\right)}{r(j)}$ $k = 0$

j =

n =

A =

	0 (0^n)	1 (1^{n-1})	2 (2^{n-1})	3 (3^{n-1})	4 (4^{n-1})
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	3	4
3	0	1	4	9	16
4	0	1	8	27	64
5	0	1	16	81	256
6	0	1	32	243	1024
7	0	1	64	729	4096
8	0	1	128	2187	16384
9	0	1	256	6561	65536
10	0	1	512	19683	$2.62144 \cdot 10^5$
11	0	1	1024	59049	$1.048576 \cdot 10^6$
12	0	1	2048	$1.77147 \cdot 10^5$	$4.194304 \cdot 10^6$
13	0	1	4096	$5.31441 \cdot 10^5$	$1.677722 \cdot 10^7$
14	0	1	8192	$1.594323 \cdot 10^6$	$6.710886 \cdot 10^7$
15	0	1	16384	$4.782969 \cdot 10^6$	$2.684355 \cdot 10^8$
16	0	1	32768	$1.434891 \cdot 10^7$	$1.073742 \cdot 10^9$
17	0	1	65536	$4.304672 \cdot 10^7$	$4.294967 \cdot 10^9$
18	0	1	$1.31072 \cdot 10^5$	$1.291402 \cdot 10^8$	$1.717987 \cdot 10^{10}$

RECURSION FORMULAE TO EXPLICIT FORMULAE

A(n+1) = j A(n) + k A(n-1)

Case of unequal roots
and initial values 0, 1

$j := 0, 1 .. 9$ $k := -1$ $n := 0, 1 .. 12$ NUMBER SEQUENCE

$r(j) := \sqrt{j^2 + 4 \cdot k}$

$p(j) := \frac{(j + r(j))}{2}$ $q(j) := \frac{(j - r(j))}{2}$

a b j k
0 1 j -1

$A_{n,j} := \frac{\left(p(j)^n - q(j)^n\right)}{r(j)}$

k = -1 shiri √5 Stabi Riban

j =

n =

A =

	0	1	2	3	4
0	0	0	0	0	0
1	1	1	0	1	1
2	0	1	0	3	4
3	-1	0	0	8	15
4	0	-1	0	21	56
5	1	-1	0	55	209
6	0	0	0	144	780
7	-1	1	0	377	2911
8	0	1	0	987	10864
9	1	0	0	2584	40545
10	0	-1	0	6765	$1.51316 \cdot 10^5$
11	-1	-1	0	17711	$5.64719 \cdot 10^5$
12	0	0	0	46368	$2.10756 \cdot 10^6$

RECEXP1.MCD RECURSION FORMULAE TO EXPLICIT FORMULAE 04-02-17

A(n+1) = j A(n) + k A(n-1)

Case of unequal roots with initial values A(0)=0, A(1)=1

$j := 0, 1 .. 9$ $k := -1$ $n := 0, 1 .. 18$ NUMBER SEQUENCE

$r(j) := \sqrt{j^2 + 4 \cdot k}$

$p(j) := \frac{(j + r(j))}{2}$ $q(j) := \frac{(j - r(j))}{2}$

a b j k
0 1 j -1

$A_{n,j} := \frac{\left(p(j)^n - q(j)^n\right)}{r(j)}$ $k = -1$

j =

Ramanujan

n =

A =

	4	5	6	7	8
0	0	0	0	0	0
1	1	1	1	1	1
2	4	5	6	7	8
3	15	24	35	48	63
4	56	115	204	329	496
5	209	551	1189	2255	3905
6	780	2640	6930	15456	30744
7	2911	12649	40391	$1.05937 \cdot 10^{5}$	$2.42047 \cdot 10^{5}$
8	10864	60605	$2.35416 \cdot 10^{5}$	$7.26103 \cdot 10^{5}$	$1.905632 \cdot 10^{6}$
9	40545	$2.90376 \cdot 10^{5}$	$1.372105 \cdot 10^{6}$	$4.976784 \cdot 10^{6}$	$1.500301 \cdot 10^{7}$
10	$1.51316 \cdot 10^{5}$	$1.391275 \cdot 10^{6}$	$7.997214 \cdot 10^{6}$	$3.411139 \cdot 10^{7}$	$1.181184 \cdot 10^{8}$
11	$5.64719 \cdot 10^{5}$	$6.665999 \cdot 10^{6}$	$4.661118 \cdot 10^{7}$	$2.338029 \cdot 10^{8}$	$9.299445 \cdot 10^{8}$
12	$2.10756 \cdot 10^{6}$	$3.193872 \cdot 10^{7}$	$2.716699 \cdot 10^{8}$	$1.602509 \cdot 10^{9}$	$7.321438 \cdot 10^{9}$
13	$7.865521 \cdot 10^{6}$	$1.530276 \cdot 10^{8}$	$1.583408 \cdot 10^{9}$	$1.098376 \cdot 10^{10}$	$5.764156 \cdot 10^{10}$
14	$2.935452 \cdot 10^{7}$	$7.331993 \cdot 10^{8}$	$9.228778 \cdot 10^{9}$	$7.528381 \cdot 10^{10}$	$4.53811 \cdot 10^{11}$
15	$1.095526 \cdot 10^{8}$	$3.512969 \cdot 10^{9}$	$5.378926 \cdot 10^{10}$	$5.160029 \cdot 10^{11}$	$3.572847 \cdot 10^{12}$
16	$4.088558 \cdot 10^{8}$	$1.683164 \cdot 10^{10}$	$3.135068 \cdot 10^{11}$	$3.536737 \cdot 10^{12}$	$2.812896 \cdot 10^{13}$
17	$1.525871 \cdot 10^{9}$	$8.064526 \cdot 10^{10}$	$1.827251 \cdot 10^{12}$	$2.424115 \cdot 10^{13}$	$2.214588 \cdot 10^{14}$
18	$5.694626 \cdot 10^{9}$	$3.863946 \cdot 10^{11}$	$1.065 \cdot 10^{13}$	$1.661513 \cdot 10^{14}$	$1.743542 \cdot 10^{15}$

RECEXP1.MCD

RECURSION FORMULAE TO EXPLICIT FORMULAE

04-02-17

A(n+1) = j A(n) + k A(n-1)

Case of unequal roots
and initial values 0, 1

$j := 0, 1 .. 9$ $k := -2$ $n := 0, 1 .. 16$ NUMBER SEQUENCE

$r(j) := \sqrt{j^2 + 4 \cdot k}$

$p(j) := \frac{(j + r(j))}{2}$ $q(j) := \frac{(j - r(j))}{2}$

a b j k
0 1 j -2

$A_{n,j} := \frac{\left(p(j)^n - q(j)^n\right)}{r(j)}$ $k = -2$

j = $2^n - 1$

n =

A =

	0	1	2	3	4
0	0	0	0	0	0
1	1	1	1	1	1
2	0	1	2	3	4
3	-2	-1	2	7	14
4	0	-3	0	15	48
5	4	-1	-4	31	164
6	0	5	-8	63	560
7	-8	7	-8	127	1912
8	0	-3	0	255	6528
9	16	-17	16	511	22288
10	0	-11	32	1023	76096
11	-32	23	32	2047	$2.59808 \cdot 10^5$
12	0	45	0	4095	$8.8704 \cdot 10^5$
13	64	-1	-64	8191	$3.028544 \cdot 10^6$
14	0	-91	-128	16383	$1.03401 \cdot 10^7$
15	-128	-89	-128	32767	$3.53033 \cdot 10^7$
16	0	93	0	65535	$1.20533 \cdot 10^8$

RECURSION FORMULAE TO EXPLICIT FORMULAE

A(n+1) = j A(n) + k A(n-1)

Case of unequal roots
and initial values 0, 1

$j := 0, 1 .. 9$ $k := -3$ $n := 0, 1 .. 18$ NUMBER SEQUENCE

$r(j) := \sqrt{j^2 + 4 \cdot k}$

$p(j) := \frac{(j + r(j))}{2}$ $q(j) := \frac{(j - r(j))}{2}$

$A_{n,j} := \frac{\left(p(j)^n - q(j)^n\right)}{r(j)}$ $k = -3$

a b j k
0, 1, j, -3

j =

n =

A =

	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	1	2	3
3	-3	-2	1	6
4	0	-5	-4	9
5	9	1	-11	9
6	0	16	-10	$2.759251 \cdot 10^{-15}$
7	-27	13	13	-27
8	0	-35	56	-81
9	81	-74	73	-162
10	0	31	-22	-243
11	-243	253	-263	-243
12	0	160	-460	$-2.510107 \cdot 10^{-13}$
13	729	-599	-131	729
14	0	-1079	1118	2187
15	-2187	718	2629	4374
16	0	3955	1904	6561
17	6561	1801	-4079	6561
18	0	-10064	-13870	$8.859986 \cdot 10^{-12}$

Equal roots when V = 0

negative j's ?

Table of values of $\sqrt{k_1^2 + 4k_2}$

$p = \frac{k_1 + V}{2}$ $q = \frac{k_1 - V}{2}$

j j

k k

k_2 \ k_1	0	1	2	3	4	5	6	7	8	9	10
+7	$2\sqrt{7}$	$\sqrt{29}$	• $2\sqrt{8}$	$\sqrt{37}$	$2\sqrt{11}$	$\sqrt{53}$	8	$\sqrt{77}$	$2\sqrt{23}$	$\sqrt{109}$	$2\sqrt{32}$
+6	$2\sqrt{6}$	5	$2\sqrt{7}$	$\sqrt{33}$	$2\sqrt{10}$	7	$2\sqrt{15}$	$\sqrt{73}$	$2\sqrt{22}$	$\sqrt{105}$	$2\sqrt{31}$
+5	$2\sqrt{5}$	$\sqrt{21}$	$2\sqrt{6}$	$\sqrt{29}$	6	$\sqrt{45}$	$2\sqrt{14}$	$\sqrt{69}$	$2\sqrt{21}$	$\sqrt{101}$	$2\sqrt{30}$
+4	4	$\sqrt{17}$	$2\sqrt{5}$	5	• $2\sqrt{8}$	$\sqrt{41}$	$2\sqrt{13}$	$\sqrt{65}$	$2\sqrt{20}$	$\sqrt{97}$	$2\sqrt{29}$
+3	$2\sqrt{3}$	$\sqrt{13}$	4	$\sqrt{21}$	$2\sqrt{7}$	$\sqrt{37}$	$2\sqrt{12}$	$\sqrt{61}$	$2\sqrt{19}$	$\sqrt{93}$	$2\sqrt{28}$
+2	$2\sqrt{2}$	3	$2\sqrt{3}$	$\sqrt{17}$	$2\sqrt{6}$	$\sqrt{33}$	$2\sqrt{11}$	$\sqrt{57}$	$2\sqrt{18}$	$\sqrt{89}$	$2\sqrt{27}$
+1	2	FIBON F • $\sqrt{5}$	$2\sqrt{2}$	$\sqrt{13}$	$2\sqrt{5}$	$\sqrt{29}$	$2\sqrt{10}$	$\sqrt{53}$	$2\sqrt{17}$	$\sqrt{85}$	$2\sqrt{26}$
0	0	1	2	3	4	5	6	7	8	9	10
−1	i	i	0	• $\sqrt{5}$	$2\sqrt{3}$	$\sqrt{21}$	RAMAN R • $2\sqrt{8}$	$\sqrt{45}$	$2\sqrt{15}$	$\sqrt{77}$	$2\sqrt{14}$
−2	i	i	i	1	$2\sqrt{2}$	$\sqrt{17}$	$2\sqrt{7}$	$\sqrt{41}$	$2\sqrt{14}$	$\sqrt{73}$	$2\sqrt{23}$
−3	i	i	i	i	2	$\sqrt{13}$	$2\sqrt{6}$	$\sqrt{37}$	$2\sqrt{13}$	$\sqrt{69}$	$2\sqrt{22}$
−4	i	i	i	i	0	3	$2\sqrt{5}$	$\sqrt{33}$	$2\sqrt{12}$	$\sqrt{65}$	$2\sqrt{21}$
−5	i	i	i	i	i	• $\sqrt{5}$	4	$\sqrt{29}$	$2\sqrt{11}$	$\sqrt{61}$	$2\sqrt{20}$
−6	i	i	i	i	i	1	$2\sqrt{3}$	5	$2\sqrt{10}$	$\sqrt{57}$	$2\sqrt{19}$
−7	i	i	i	i	i	i	$2\sqrt{2}$	$\sqrt{21}$	6	$\sqrt{53}$	$2\sqrt{18}$
−8	i	i	i	i	i	i	2	$\sqrt{17}$	• $2\sqrt{8}$	7	$2\sqrt{17}$
−9	i	i	i	i	i	i	0	$\sqrt{13}$	$2\sqrt{7}$	$\sqrt{45}$	8
−10	i	i	i	i	i	i	i	3	$2\sqrt{6}$	$\sqrt{41}$	$2\sqrt{15}$
−11	i	i	i	i	i	i	i	• $\sqrt{5}$	$2\sqrt{5}$	$\sqrt{37}$	$2\sqrt{14}$

1 4

i $2\sqrt{3}$

RECEXP4.MCD

RECURSION FORMULAE TO EXPLICIT FORMULAE

04-02-19

A(n+1) = j A(n) + k A(n-1)

$j := 0, 1 .. 15$ $k := -9, -8 .. 9$ Squares of radical values

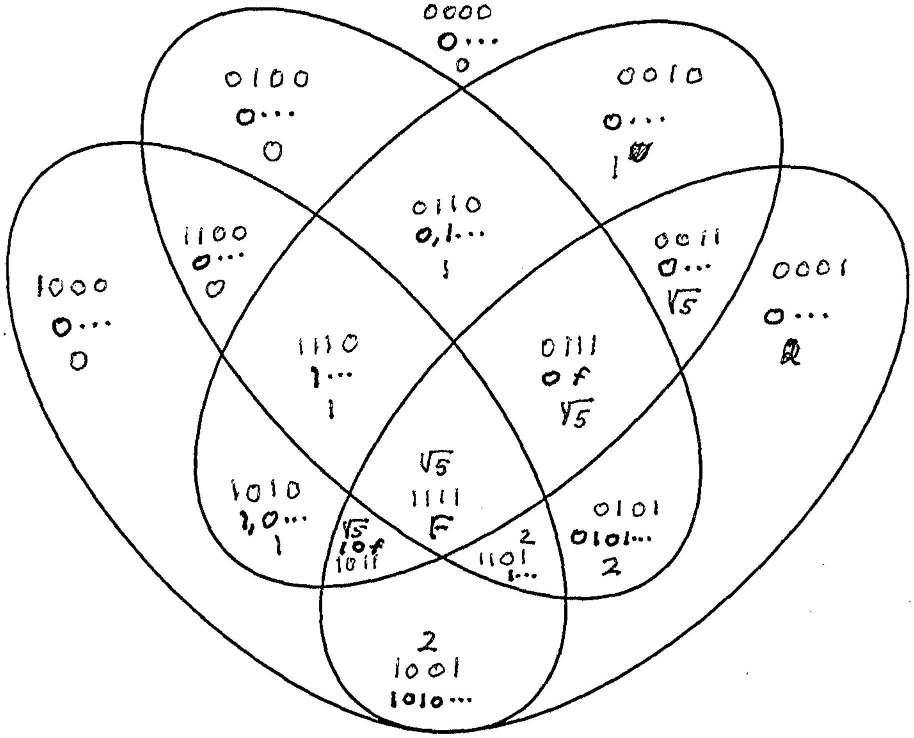
$s_{k,j} := j^2 + 4 \cdot k$

j

k

s =

	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
-9	0	-36	-35	-32	-27	-20	-11	0	13	28	45	64	85	108	133	160	189
-8	0	-32	-31	-28	-23	-16	-7	4	17	32	49	68	89	112	137	164	193
-7	0	-28	-27	-24	-19	-12	-3	8	21	36	53	72	93	116	141	168	197
-6	0	-24	-23	-20	-15	-8	1	12	25	40	57	76	97	120	145	172	201
-5	0	-20	-19	-16	-11	-4	5	16	29	44	61	80	101	124	149	176	205
-4	0	-16	-15	-12	-7	0	9	20	33	48	65	84	105	128	153	180	209
-3	0	-12	-11	-8	-3	4	13	24	37	52	69	88	109	132	157	184	213
-2	0	-8	-7	-4	1	8	17	28	41	56	73	92	113	136	161	188	217
-1	0	-4	-3	0	5	12	21	32	45	60	77	96	117	140	165	192	221
0	0	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
1	0	4	5	8	13	20	29	40	53	68	85	104	125	148	173	200	229
2	0	8	9	12	17	24	33	44	57	72	89	108	129	152	177	204	233
3	0	12	13	16	21	28	37	48	61	76	93	112	133	156	181	208	237
4	0	16	17	20	25	32	41	52	65	80	97	116	137	160	185	212	241
5	0	20	21	24	29	36	45	56	69	84	101	120	141	164	189	216	245
6	0	24	25	28	33	40	49	60	73	88	105	124	145	168	193	220	249
7	0	28	29	32	37	44	53	64	77	92	109	128	149	172	197	224	253
8	0	32	33	36	41	48	57	68	81	96	113	132	153	176	201	228	257
9	0	36	37	40	45	52	61	72	85	100	117	136	157	180	205	232	261



$2^4 = 16$

A_0 A_1 j k

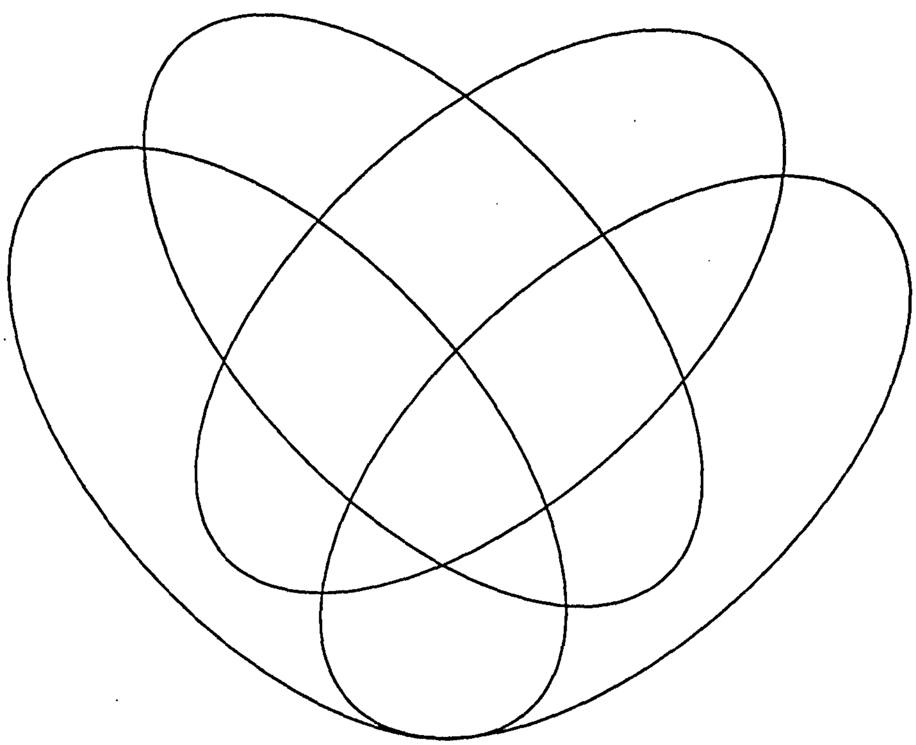
layers	
0	1
1	4
2	6
3	4
4	1

red = sequence

green = $\sqrt{j^2+4k}$

layers	
0	0
1	2 0 0 ϕ
2	√5 2 2 1 1 0
3	√5 √5 2 1
4	√5

F = Fibonacci



PYTHAGOREAN TRIPLES

$m := 1, 2 .. 8$ $\qquad$ $n := 1, 2 .. 10$

$A_{m,n} := 2 \cdot m \cdot n$ $\qquad$ $B_{m,n} := m^2 - n^2$ $\qquad$ $C_{m,n} := m^2 + n^2$

n

m

B =

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-3	-8	-15	-24	-35	-48	-63	-80	-99
2	0	3	0	-5	-12	-21	-32	-45	-60	-77	-96
3	0	8	5	0	-7	-16	-27	-40	-55	-72	-91
4	0	15	12	7	0	-9	-20	-33	-48	-65	-84
5	0	24	21	16	9	0	-11	-24	-39	-56	-75
6	0	35	32	27	20	11	0	-13	-28	-45	-64
7	0	48	45	40	33	24	13	0	-15	-32	-51
8	0	63	60	55	48	39	28	15	0	-17	-36

A =

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	4	6	8	10	12	14	16	18	20
2	0	4	8	12	16	20	24	28	32	36	40
3	0	6	12	18	24	30	36	42	48	54	60
4	0	8	16	24	32	40	48	56	64	72	80
5	0	10	20	30	40	50	60	70	80	90	100
6	0	12	24	36	48	60	72	84	96	108	120
7	0	14	28	42	56	70	84	98	112	126	140
8	0	16	32	48	64	80	96	112	128	144	160

C =

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	5	10	17	26	37	50	65	82	101
2	0	5	8	13	20	29	40	53	68	85	104
3	0	10	13	18	25	34	45	58	73	90	109
4	0	17	20	25	32	41	52	65	80	97	116
5	0	26	29	34	41	50	61	74	89	106	125
6	0	37	40	45	52	61	72	85	100	117	136
7	0	50	53	58	65	74	85	98	113	130	149
8	0	65	68	73	80	89	100	113	128	145	164

PYTHAGOREAN TRIPLES

$m > n > 0$ positive integer

$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2$

then $a^2 + b^2 = c^2$

Since $4m^2n^2 + (m^4 - 2m^2n^2 + n^4) = m^4 + 2m^2m^2 + n^4$
$= (m^2 + n^2)^2$.

Examples:

$m = 2$, $n = 1$: $a = 4$, $b = 3$, $c = 5$

$m = 3$, $n = 1$: $a = 6$, $b = 8$, $c = 10$

$m = 3$, $n = 2$: $a = 12$, $b = 5$, $c = 13$

$m = 4$, $n = 1$: $a = 8$, $b = 15$, $c = 17$

Special case I

Pick an odd positive integer

e.g. 7

square it

49,

~~divide~~ Separate it into 2 as near equal as possible part

$49 = 24 + 25$ $\qquad 49 = 25 + 24$

$49 = 25^2 - 24^2$

7, 24, 25 are triples

then $7^2 + 24^2 = 25^2$

$49 + 576 = 625$

$m = 2$
$n = 3$
$a = 12,\ b = -5,\ c = 13$

Pick an even positive integer, h

then $2h,\ h^2 - 1,\ h^2 + 1$ are triple

e.g. $h = 4$

$a = 8$ $\quad a^2 = 64$
$b = 15$ $\quad b^2 = 225$
$c = 17$ $\quad c^2 = 289$

8, 0

5, 2
$a = 20$
$b = 21$
$c = 29$

4, 3
$a = 24$
$b = 7$
$c = 25$

Make Table

m (across), n (down)

n \ m	2	3	4	5	
1	3, 4, 5	6, 8, 10	8, 15, 17		
2	0, 0, 8	5, 12, 13		20, 21, 29	
3	−5, 12, 13		7, 24, 25		

SEQUENCES

SUMMATION of SERIES

TWO "TEXT BOOK" METHODS

I The [Gauss] Inversion method:

This method can be used when the differences between successive terms are constant. $(=\Delta)$

$$S_n = I + (I+\Delta) + (I+2\Delta) + (I+3\Delta) + \cdots + (I+(N-1)\Delta) + \text{~~}(I+N\Delta)\text{~~}$$

(N^{th})

$$S_n = \text{~~}(I+N\Delta)\text{~~} + (I+(N-1)\Delta) + (I+(N-2)\Delta) + \cdots + (I+\Delta) + I$$

Adding: where I is the value of the first term

$$2S_n = \text{~~}(2I+N\Delta)(N\ldots)\text{ or } S =\text{~~} \quad [2I+(N-1)\Delta]N \qquad S_n = \frac{N[N\cdot\Delta - \Delta + 2I]}{2}$$

e.g. If $I=1$, $\Delta = 1$

$S_n = \frac{N(N+1)}{2}$, the sum of the first N, integers

$$\text{cf} \quad V = \frac{(n+R)!}{(n-1)!(R+1)!}$$

II Power Series Method;

This method can be used with a power series with all terms having an identical ~~constant~~ coeficients

$$S_n = ap^0 + ap^1 + ap^2 + ap^3 + \cdots + ap^{n-1}$$

(n^{th})

$$p\cdot S_n = ap^1 + ap^2 + ap^3 \cdots + \cdots ap^n + \text{~~}ap^{n+1}\text{~~}$$

$$pS_n - S_n = ap^{n} - ap^0 \quad \text{or} \quad S_n = a\frac{p^{n}-1}{p-1}$$

eg $a=1$, $p=2$

$$S_n = 2^{n} - 1 \checkmark$$

$$\text{cf} \quad \frac{p^n - q^n}{p-q}$$

SOME BASIC PATTERNS

The 2^n series:

Has in common with the Fibonacci Series

Δ's all =

In obverse Yanghuis, the values of 2^n are like Maxwell's singular points

An exclusion or homogenization occurs at values, 2, 4, 8, 16 ...

The $A_{n+2} = 10(A_{n+1} - A_n)$ recursion formula

leads to values very close to those

related to the fundamental physical constants, α, μ, S

and the Planck particle.

i.e. Both $F_{n+2} = F_{n+1} + F_n$ and $A_{n+2} = 10A_{n+1} - 10A_n$ } are basic in nature

The obverse yanghuis are related to Wolfram's fractal class

[Wolfram's 4 patterns: Uniform, Fractal, Random, Embedded]

$\exists$ a certain amount of information

in the form of the a recursion relation such as

$A_{n+2} = JA_{n+1} + KA_n$ Call this "primary" information W_0

Then for all resulting sequences

The additional information of 4 numbers - a, b, J, K covers the set $\{W_0\}$

where a and b are initial values

$N := 1, 2 .. 20$

DIAPHANT

FOR WHAT N and M [integers] is S = P

How about $N^2 = M^3$

$$S(N) := N \cdot \frac{(N+1)}{2} \qquad P(N) := N \cdot (N+1) \cdot \frac{(N+2)}{6}$$

N =	S(N) =	P(N) =	S(N+100) =	S(N+40) =	P(N+20) =	S(N+60) =
1	1	1	5151	861	1771	1891
2	3	4	5253	903	2024	1953
3	6	10	5356	946	2300	2016
4	10	20	5460	990	2600	2080
5	15	35	5565	1035	2925	2145
6	21	56	5671	1081	3276	2211
7	28	84	5778	1128	3654	2278
8	36	120	5886	1176	4060	2346
9	45	165	5995	1225	4495	2415
10	55	220	6105	1275	4960	2485
11	66	286	6216	1326	5456	2556
12	78	364	6328	1378	5984	2628
13	91	455	6441	1431	6545	2701
14	105	560	6555	1485	7140	2775
15	120	680	6670	1540	7770	2850
16	136	816	6786	1596	8436	2926
17	153	969	6903	1653	9139	3003
18	171	1140	7021	1711	9880	3081
19	190	1330	7140	1770	10660	3160
20	210	1540	7260	1830	11480	3240

	N^2	N^3
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	243
8	64	512

S = P

N	S	N	P
1	1	1	1
4	10	3	10
15	120	8	120
55	1540	20	1540
119	7140	34	7140

Next > 51,000
70,000
150,000

As the gaps widen the probability for coincidence decreases
i.e. the coincidence become rare

Harmonics

N=4 octave

N=7 84 : 28 12 : 4
30 G

N=10 2 octaves

N=13 455 : 91 5 : 1
5° E

We perceive the world of the low values of N

The N=4
10 - 20 Harmonic
N=5
3 : 7
3 : 8

P	S
1	1
10	10

occasionally 120 120

$N := 1, 2 .. 20$

DIAPHANT

Difference of 1

$S(N) := N \cdot \frac{(N+1)}{2}$ $P(N) := N \cdot (N+1) \cdot \frac{(N+2)}{6}$

N =	S(N) =	P(N) =	S(N+20) =	S(N+40) =	P(N+20) =
1	1	1	231	861	1771
2	3	4	253	903	2024
3	6	10	276	946	2300
4	10	20	300	990	2600
5	15	35	325	1035	2925
6	21	56	351	1081	3276
7	28	84	378	1128	3654
8	36	120	406	1176	4060
9	45	165	435	1225	4495
10	55	220	465	1275	4960
11	66	286	496	1326	5456
12	78	364	528	1378	5984
13	91	455	561	1431	6545
14	105	560	595	1485	7140
15	120	680	630	1540	7770
16	136	816	666	1596	8436
17	153	969	703	1653	9139
18	171	1140	741	1711	9880
19	190	1330	780	1770	10660
20	210	1540	820	1830	11480

36

$S - P = \pm 1$

	N	S	N	P	δ
	2	3	2	4	2
4	6	21	4	20	2
4	10	55	6	56	8
23	33	561	14	560	7
26	59	1770	21	1771	4
17	76	2926	25	2925	

$P_N = P_{N-1} + S_N$

$P_N = P_{N-1} + \frac{N(N+1)}{2}$

$P_N - P_{N-2} = N^2$

$S_N - S_{N-5} = 5(N-2)$

Next

$S - P = \pm 10$

04-02-29

SLOPES OF SEQUENCES ($p=1$)

$\Delta P = 2$	$\Delta P = 2$	$\Delta P = 2$	$\Delta P = 2$
$\Delta K = 2$	$\Delta K = 4$	$\Delta K = 6$	$\Delta P = 4$
$\Delta B = 5$	$\Delta B = 13$	$\Delta B = 25$	$\Delta K = 41$

$\lambda =$ slope $\frac{\Delta K}{\Delta P} = 1$ 2   3   4

$$\Delta B = 2\lambda^2 + 2\lambda + 1 = \boxed{\lambda^2 + (\lambda+1)^2 = \Delta B}$$

$$\Delta B = 2\left(\frac{\Delta K}{2}\right)^2 + 2\frac{\Delta K}{2} + 1 = \frac{1}{2}\Delta K^2 + \Delta K + 1$$

$P=4$ SEQUENCES

B	P	K	ΔB	ΔP	ΔK	λ
15	4	5	17	4	6	3/2
32	8	11				
19	4	6	17	4	6	3/2
36	8	12				
34	4	9	37	4	10	5/2
71	8	19				
40	4	10	37	4	10	5/2
77	8	20				

~~$\frac{1}{2}$ 36 + 6 + 1~~ not yet 25

$$\frac{9}{4} + \frac{25}{4} = \frac{34}{4} = \frac{17}{2}$$

Try

$$\boxed{\Delta B = 2\left[\lambda^2 + (\lambda+1)^2\right]}$$

$$2\left[\frac{25}{4} + \frac{49}{4}\right] \qquad \frac{74}{2} = 37 \checkmark$$

DOES $P=1 \Rightarrow$ all odd P's ?

SCRAPS 2004

1.	APHISTRY.WPD	04/01/02	SOME APHORISMS RE HISTORY
2.	RAMFORM.WPD	04/01/01	RE RAMANUJAN NUMBERS
3.	Raman.mcd	04/01/04	RAMANUJAN FORMULAE
4.	Fibon1.mcd	04/01/04	FIBONACCI FORMULAE
5.	EVOSTEP.WPD	04/01/05	THE DISCRETENESS OF CHANGE
6.	FIBON2.MCD	04/01/06	ALTERNATE FIBONACCI NUMBERS
7.	FIBON3.MCD	04/01/06	SUBSET OF FIBONACCI SEQUENCE
8.			
9.			
10.			

04 - 02 - 23

P = 1
THE "SQUARES" FAMILY

The sequence of squares, 1, 4, 9, 16, 25, ... $S(s) = s^2$, = , The B_1 sequence is a "seed" sequence. $S = s^2$, $s = 1, 2, 3, 4, 5$ ...
Each number of the sequence is an inial number for anoth sequence.

[By pascal]

The b = a+1 family

	$S: s^2 = B_1$	ΔB	K = aP a	J = bP b
~~S = 2~~	~~$S^2 = 4$~~	~~5~~	~~1~~	~~2~~
s = 2	S = 4	5	1	2
3	9	13	2	3
4	16	25	3	4
5	25	41	4	5
s	s^2	$s^2+(s-1)^2$	s-1	s

The sequence P(n) = odd number
$P(n) = 2n-1 \quad n = 1, 2, \ldots$

$J = P + K$

KBP

Formulae I for this "square base" family

$B_1 = s^2, \quad \Delta B(s) = s^2 + (s-1)^2, \quad a(s) = s-1, \quad b(s) = s$

$P(n) = 2n-1, \quad K(s,n) = (s-1)(2n-1), \quad J(s,n) = s(2n-1)$

$B(s,n) = ns^2 + (n-1)(s-1)^2$ [Explicit] $B(s,n) = B_1 + (n-1)\Delta B$ (~~$n\Delta B$~~)

RECURSION

~~B(2, n) = 4 + 5n n = 0, 1,~~

$B(2, n) = 4 + 5(n-1) \quad n = 1, 2, \ldots$

$B(3, n) = 9 + 13(n-1)$

$B(4, n) = 16 + 25(n-1)$

FORMULA II

$B(s,n) = ns^2 + (n-1)[s^2 + (s+1)^2]$

$= ns^2 + (n-1)(s+1)^2$

The ΔB sequence: 5, 13, 25, 41, 61, 85

Δ: 8 12 16 20 24

Δ^2: 4 4 4 4

The K sequence

There may be more differentia?

e.g. $6 + 5n$ $B_1 = 1$

$4 + 13n$ $B_1 = 4$

FORMULAE I and II

account for all the "square derived" i.e. P = 1 fulcrum numbers (~~derived~~)

i.e. seeds at P = 1

$B(P, K) =$

$K = s-1$ ~~$K = s-1$~~ $B(s,n) = ns^2 + (n-1)(s\pm1)^2$?

$s = K+1$ $P = 2n-1$

$n = \frac{P+1}{2}$ $\Delta B(K) = K^2 + (K+1)^2$ $\Delta B(K) = K^2 + (K-1)^2$

$B_1(K) = (K+1)^2$

Associate ~~K with~~

SEQUENCE RECURSIVES POSSIBLE

(allow in addition 0,1 and more)

4 9 14

FULCRUMS — TWO PARAMETER EXPLICIT

$B(s, n)$ ~~w~~ $B(P, K)$ $P, K = 1, 2, \ldots$ all

subset

$B(s,n) = ns^2 + (n-1)(s\pm1)^2$ (int) $\left[B(P,K) = K + \frac{K^2+K}{P} + \frac{P+1}{2} \right]$

$P = 1, 2, 3$

$K = 0, 1, 2, 3$

~~(B)~~ = s n P K

$+ \quad 16 = B(1,4)$ $+25 = B(5,1)$ $16 = B(1,3)$ $25 = (1,4)$ $36 = (1,5)$

$- \quad 16 = B(4,1)$ $-25 = B(5,1)$ $16 = B(7,6)$ $36 = (8,12)$

$+ \quad 84 = B(3,4)$ $+98 = B(4,3)$

$- \quad 48 = B(3,4)$ $-66 = B(4,3)$

$5 + 30 + 1 = 36$

2 4/0

9/0

16/0

42 (only 1, 25!)

$12 + \frac{144+12}{8} + \frac{9}{2}$

$+ \quad 109 = B(3,5)$

$- \quad 61 = B(3,5)$

$\frac{156}{8} = \frac{39}{2}$

$+36 = B(6,1)$

$-36 =$

$12 + \frac{39+9}{2}$ $24 = 36$

$3 + \frac{12}{2} + \frac{3}{2}$

$P = 0$

~~$K = P$~~

$5 + \frac{25+5}{4} + \frac{5}{2}$

25

$\frac{15}{2} + \frac{5}{2}$

$K + (K^2 + K)$

$P = 1, 2, 3 \ldots$

$K = 0, 1, 2, 3$

K=0	P	B
	1	1
	2	—
	3	2 X

K=0	P=1	B=1
	2	—
	3	2
	4	—
	5	3

K=1	P=	B
	1	4

03-08-21

General Eq. for odd P: $2B = (2N^2 + 2N + 1) \cdot P \pm (2N+1)$

$P = 1$ $\quad 2B = (2N^2 + 2N + 1) \pm (2N+1)$

$+ \quad B = N^2 + 2N + 1 \quad\quad B = (N+1)^2$ ~~all~~ squares for all N

$- \quad B = N^2 \quad\quad B = N^2$

$P = 3 \quad + \quad B = 3N^2 + 4N + 2$

$- \quad B = 3N^2 + 2N + 1$

$N = 1 \quad + \quad B = 9$ a square → { K = 3, J = 6 ; K = 2, J = 3 } double Σ = 33, Σ = 21

$- \quad B = 6$ [an L = 1] K = 2 J = 5 Σ = 15

$N = 2 \quad + \quad B = 23$ K = 6, J = 9 Σ = 153

$- \quad B = 17$ K = 5, J = 8 Σ = 100

$N = 3 \quad + \quad B = 41$ K = 10, J = 14 Σ = 469 (?, but P = 3)

$- \quad B = 34$

• $N = 4 \quad + \quad B = 66$

$- \quad B = 57$

$P = 5 \quad + \quad B = 5N^2 + 6N + 3$

$- \quad B = 5N^2 + 4N + 2$

$N = 1 \quad + \quad B = 14$

$- \quad B = 11$ an

$N = 2 \quad + \quad B = 35$ [an L = 5]

$- \quad B = 30$

• $N = 3 \quad + \quad B = 66$

$- \quad B = 59$

$N = 4 \quad + \quad B = 107$

$- \quad B = 98$

$P = 7 \quad + \quad B = 7N^2 + 8N + 4$

$- \quad B = 7N^2 + 6N + 3$

$N = 1 \quad + \quad B = 19$

$- \quad B = 16$ a square → { K = ; K = }

4 148
139

$N = 2 \quad + \quad B = 48$

$- \quad B = 43$

$N = 3 \quad + \quad B = 91$

$- \quad B = 84$

Questions: Given B what is K and J?

When will a B have two solutions?

When will B(P, N) take on same values? or more


p7 #
® 2
$Ç $Ç $Ç
$ 1
\ \ $™ $™ $™ $§

$P=1 \quad +B=(N+1)^2$

$\quad\quad -B = N^2$

Do all squares have >1 solution no

$N=1$	$+B=4$	$K=1,\ J=2$	$\Sigma=5$
	$-B=1$	— —	
$N=2$	$+B=9$	$K=2,\ J=3$	$\Sigma=21$
	$-B=4$	~~as above~~ no second	
$N=3$	$+B=16$	$K=3,\ J=4$	$\Sigma=54$
	$-B=9$	also ~~as above~~ $K=3,\ J=5$	$\Sigma=33$
$N=4$	$+B=25$	$K=4,\ J=5$	$\Sigma=110$
	$-B=16$	also ~~as above~~ $K=6,\ J=13$	$\Sigma=117$
$N=5$	$+B=36$	$K=5,\ J=6$	$\Sigma=195$
? →	$-B=25$	no second?	—
$N=6$	$+B=49$	$K=6,\ J=7$	$\Sigma=315$
	$-B=36$	also $K=12,\ J=20$	$\Sigma=510$
$N=7$	$+B=64$	$K=7,\ J=8$	$\Sigma=476$
	$-B=49$	$K=17,\ J=29$	$\Sigma=986$
$N=8$	$+B=81$	$K=8,\ J=9$	$\Sigma=684$
	$-B=64$		

$P=1$ The Squares

$B(n) = n^2 \quad 1, 4, 9, 16, \ldots$

B J K P

//

Ge 11:32 And the days of Terah were two hundred and five years: and Terah died in Haran.

03-10-12

First of all ∃ B's and not B's
a dichotomy, dyad, as ∃ P's and not P's

Tests for B

- The existence of two sigmas that are equal

The Ramanujan case: $\sum_{1}^{B-1} U = \sum_{B} A$

$S(U) = N \cdot B - \frac{N(N+1)}{2}$

$S(A) = N \cdot B + \frac{N(N+1)}{2}$

We divide the Universe (of numbers) into dyads

$B = 4 + 5N$

B	N
4	0
9	1
14	2
19	3

B =

$14 = 9 + 5$

$B_N = B_{N-1} + 5$

k = 1

Species of Sequences

I $a + nb$ recursion $B_{n+1} = B_n + b$ e.g. $6 = 1 + n \cdot 5$

II

II $B_{n+1} = k B_n + B_{n-1}$ Ramanujan #'s

Explorations

1) $B_{n+1} = a B_n \pm B_{n-1}$ Ramanujan $a = 6$ $B_0 = 0,\ B_1 = 1,\ B_2 = 6$

2) $a = 2$

$11 = 2 \cdot 6 - 1$ $B_0 = 1,\ B_1 = 6$

selects sub-sequence →

$21 = 2 \cdot 11 - 1$

$41 = 2 \cdot 21 - 1$

$81 = 2 \cdot 41 - 1$

all are B's but does the sequence continue?

3)

$a = 2$ $9 = 2 \cdot 4 + 1$ $B_0 = 4,\ B_1 = 4$

$22 = 2 \cdot 9 + 4$

$53 = 2 \cdot 22 + 9$

$128 = 2 \cdot 53 + 22$

$13 = 2 \cdot 6 + 1$

$32 = 2 \cdot 13 + 6$

$77 = 2 \cdot 32 + 13$

✓ $11 = 2 \cdot 6 - 1$ $B_1 = 6,\ B_0 = 1$

✓ $16 = 2 \cdot 11 - 6$

✓ $21 = 2 \cdot 16 - 11$ A = 2

✓ $26 = 2 \cdot 21 - 16$

A = 5 ✓ $31 = 2 \cdot 26 - 21$

✓ $36 = 2 \cdot 31 - 26$

✓ $41 = 2 \cdot 36 - 31$

✓ $46 = 2 \cdot 41 - 36$

✓ $51 = 2 \cdot 46 - 41$

✓ 56

92
41
1
102
46
6

Can we write a valid formula for all B's of the form $B_n = a B_{n-1} \pm B_{n-2}$

with selected B_1 and B_0's ?

what set of valid B_1's and B_0's ?

i.e. B's satisfy a recursion formula of the form $B_n = a B_{n-1} \pm B_{n-2}$

for $B_0 = 1,\ 0$ $B_1 = \ldots$

$K = K(P, B)$

03-09-26

FULCRUM & MCD

$$V(N) = B \cdot N - \frac{N(N+1)}{2} \;;\quad S(N) = ~~B \cdot N + \frac{N(N+1)}{2}~~ = B \cdot N + \frac{N(N+1)}{2}$$

if $V = S$ with one N: $N(N+1) = 0$ ∴ $N = 0$, or -1

But there are 2 N's, J and K

$$V(J) = B \cdot J - \frac{J(J+1)}{2} \;;\quad S(K) = B \cdot K + \frac{K(K+1)}{2}$$

$V = S$

$$2B(J-K) = J(J+1) + K(K+1)$$

$$J - K = P$$

$$2BP = (P+K)(P+K+1) + K(K+1) = P^2 + 2PK + P + 2K^2 + 2K$$

$$2BP - P^2 - P = 2(K^2 + K + PK)$$

Assign values to B and P, solve for K (and J)

Let $W = \frac{2BP - P^2 - P}{2}$

Then $K^2 + K(1+P) - W = 0$

$$K = \frac{-(1+P) + \sqrt{(1+P)^2 + 4W}}{2}$$, K must be an integer

$K = K(B, P)$

Assign P, let $B = 1, 2, \cdots$

Assign B, let $P = 1, 2, \cdots$

$$\frac{-2 + \sqrt{4 + 4W}}{2} = -1 + \sqrt{1 + 4 = (1} = \sqrt{2} - 1$$

B

P

FULCRUM5.MCD 03-09-23

$B := 110$

$N := 1, 2 .. (B - 1)$

$$S(N) := B \cdot N + \frac{(N \cdot (N+1))}{2}$$

$$V(N) := B \cdot N - \frac{(N \cdot (N+1))}{2}$$

B K J P
49 19 69 40

$J < B$ $P = J - K$

$J = B + K < B$

$K < B - P$

$J < B$

$K \leq B - P$

$P + K < B$

B

P

V(N) =	N =	S(N) =	V(N+20) =	N+20 =	S(N+20) =	V(N+40) =	N+40 =	S(N+40) =
109	1	111	$2.079 \cdot 10^3$	21	$2.541 \cdot 10^3$	$3.649 \cdot 10^3$	41	$5.371 \cdot 10^3$
217	2	223	$2.167 \cdot 10^3$	22	$2.673 \cdot 10^3$	$3.717 \cdot 10^3$	42	$5.523 \cdot 10^3$
324	3	336	$2.254 \cdot 10^3$	23	$2.806 \cdot 10^3$	$3.784 \cdot 10^3$	43	$5.676 \cdot 10^3$
430	4	450	$2.34 \cdot 10^3$	24	$2.94 \cdot 10^3$	$3.85 \cdot 10^3$	44	$5.83 \cdot 10^3$
535	5	565	$2.425 \cdot 10^3$	25	$3.075 \cdot 10^3$	$3.915 \cdot 10^3$	45	$5.985 \cdot 10^3$
639	6	681	$2.509 \cdot 10^3$	26	$3.211 \cdot 10^3$	$3.979 \cdot 10^3$	46	$6.141 \cdot 10^3$
742	7	798	$2.592 \cdot 10^3$	27	$3.348 \cdot 10^3$	$4.042 \cdot 10^3$	47	$6.298 \cdot 10^3$
844	8	916	$2.674 \cdot 10^3$	28	$3.486 \cdot 10^3$	$4.104 \cdot 10^3$	48	$6.456 \cdot 10^3$
945	9	$1.035 \cdot 10^3$	$2.755 \cdot 10^3$	29	$3.625 \cdot 10^3$	$4.165 \cdot 10^3$	49	$6.615 \cdot 10^3$
$1.045 \cdot 10^3$	10	$1.155 \cdot 10^3$	$2.835 \cdot 10^3$	30	$3.765 \cdot 10^3$	$4.225 \cdot 10^3$	50	$6.775 \cdot 10^3$
$1.144 \cdot 10^3$	11	$1.276 \cdot 10^3$	$2.914 \cdot 10^3$	31	$3.906 \cdot 10^3$	$4.284 \cdot 10^3$	51	$6.936 \cdot 10^3$
$1.242 \cdot 10^3$	12	$1.398 \cdot 10^3$	$2.992 \cdot 10^3$	32	$4.048 \cdot 10^3$	$4.342 \cdot 10^3$	52	$7.098 \cdot 10^3$
$1.339 \cdot 10^3$	13	$1.521 \cdot 10^3$	$3.069 \cdot 10^3$	33	$4.191 \cdot 10^3$	$4.399 \cdot 10^3$	53	$7.261 \cdot 10^3$
$1.435 \cdot 10^3$	14	$1.645 \cdot 10^3$	$3.145 \cdot 10^3$	34	$4.335 \cdot 10^3$	$4.455 \cdot 10^3$	54	$7.425 \cdot 10^3$
$1.53 \cdot 10^3$	15	$1.77 \cdot 10^3$	$3.22 \cdot 10^3$	35	$4.48 \cdot 10^3$	$4.51 \cdot 10^3$	55	$7.59 \cdot 10^3$
$1.624 \cdot 10^3$	16	$1.896 \cdot 10^3$	$3.294 \cdot 10^3$	36	$4.626 \cdot 10^3$	$4.564 \cdot 10^3$	56	$7.756 \cdot 10^3$
$1.717 \cdot 10^3$	17	$2.023 \cdot 10^3$	$3.367 \cdot 10^3$	37	$4.773 \cdot 10^3$	$4.617 \cdot 10^3$	57	$7.923 \cdot 10^3$
$1.809 \cdot 10^3$	18	$2.151 \cdot 10^3$	$3.439 \cdot 10^3$	38	$4.921 \cdot 10^3$	$4.669 \cdot 10^3$	58	$8.091 \cdot 10^3$
$1.9 \cdot 10^3$	19	$2.28 \cdot 10^3$	$3.51 \cdot 10^3$	39	$5.07 \cdot 10^3$	$4.72 \cdot 10^3$	59	$8.26 \cdot 10^3$
$1.99 \cdot 10^3$	20	$2.41 \cdot 10^3$	$3.58 \cdot 10^3$	40	$5.22 \cdot 10^3$	$4.77 \cdot 10^3$	60	$8.43 \cdot 10^3$

$$P = P(B, K)$$

$$B - K = \frac{K^2 + K}{P} + \frac{P+1}{2}$$

$B > K + P$

$P < B - K$

solve for $P = P(B, K)$

$$B - K = \frac{K^2 + K}{P} + \frac{P+1}{2}$$

$$2P(B-K) = 2(K^2+K) + P^2 + P$$

$$P^2 + (1 - 2B + 2K)P + 2(K^2+K) = 0$$

$$P = \frac{[2(B-K) - 1] \pm \sqrt{[2(B-K)-1]^2 - 8(K^2+K)}}{2}$$

K	B	P
1	4	4
2	6	4
2	9	12
3	9	8
4	11	8
5	14	12

H	V	P
1	4	4
2	6	4
2	9	12
3	9	8
3	16	24
4	11	8
4	25	40

9

1.2 3 4 [5] [6] 7 8

2 3 [4] 5

6 7 8 9 10 11

H / K	V / B	P
1	4	1
2	6	3
2	9	1

B	P	K
B=4	1	1
9	1	2
6	3	2
9	8	3

8 = 9 10+11+12 33

7

6

5

4

3

$B := 81$

$N := 1, 2 .. (B - 1)$

$$S(N) := B \cdot N + \frac{(N \cdot (N+1))}{2}$$

$$V(N) := B \cdot N - \frac{(N \cdot (N+1))}{2}$$

V(N) =	N =	S(N) =	V(N+20) =	N+20 =	S(N+20) =	V(N+40) =	N+40 =	S(N+40) =
80	1	82	$1.47 \cdot 10^3$	21	$1.932 \cdot 10^3$	$2.46 \cdot 10^3$	41	$4.182 \cdot 10^3$
159	2	165	$1.529 \cdot 10^3$	22	$2.035 \cdot 10^3$	$2.499 \cdot 10^3$	42	$4.305 \cdot 10^3$
237	3	249	$1.587 \cdot 10^3$	23	$2.139 \cdot 10^3$	$2.537 \cdot 10^3$	43	$4.429 \cdot 10^3$
314	4	334	$1.644 \cdot 10^3$	24	$2.244 \cdot 10^3$	$2.574 \cdot 10^3$	44	$4.554 \cdot 10^3$
390	5	420	$1.7 \cdot 10^3$	25	$2.35 \cdot 10^3$	$2.61 \cdot 10^3$	45	$4.68 \cdot 10^3$
465	6	507	$1.755 \cdot 10^3$	26	$2.457 \cdot 10^3$	$2.645 \cdot 10^3$	46	$4.807 \cdot 10^3$
539	7	595	$1.809 \cdot 10^3$	27	$2.565 \cdot 10^3$	$2.679 \cdot 10^3$	47	$4.935 \cdot 10^3$
612	8	684	$1.862 \cdot 10^3$	28	$2.674 \cdot 10^3$	$2.712 \cdot 10^3$	48	$5.064 \cdot 10^3$
684	9	774	$1.914 \cdot 10^3$	29	$2.784 \cdot 10^3$	$2.744 \cdot 10^3$	49	$5.194 \cdot 10^3$
755	10	865	$1.965 \cdot 10^3$	30	$2.895 \cdot 10^3$	$2.775 \cdot 10^3$	50	$5.325 \cdot 10^3$
825	11	957	$2.015 \cdot 10^3$	31	$3.007 \cdot 10^3$	$2.805 \cdot 10^3$	51	$5.457 \cdot 10^3$
894	12	$1.05 \cdot 10^3$	$2.064 \cdot 10^3$	32	$3.12 \cdot 10^3$	$2.834 \cdot 10^3$	52	$5.59 \cdot 10^3$
962	13	$1.144 \cdot 10^3$	$2.112 \cdot 10^3$	33	$3.234 \cdot 10^3$	$2.862 \cdot 10^3$	53	$5.724 \cdot 10^3$
$1.029 \cdot 10^3$	14	$1.239 \cdot 10^3$	$2.159 \cdot 10^3$	34	$3.349 \cdot 10^3$	$2.889 \cdot 10^3$	54	$5.859 \cdot 10^3$
$1.095 \cdot 10^3$	15	$1.335 \cdot 10^3$	$2.205 \cdot 10^3$	35	$3.465 \cdot 10^3$	$2.915 \cdot 10^3$	55	$5.995 \cdot 10^3$
$1.16 \cdot 10^3$	16	$1.432 \cdot 10^3$	$2.25 \cdot 10^3$	36	$3.582 \cdot 10^3$	$2.94 \cdot 10^3$	56	$6.132 \cdot 10^3$
$1.224 \cdot 10^3$	17	$1.53 \cdot 10^3$	$2.294 \cdot 10^3$	37	$3.7 \cdot 10^3$	$2.964 \cdot 10^3$	57	$6.27 \cdot 10^3$
$1.287 \cdot 10^3$	18	$1.629 \cdot 10^3$	$2.337 \cdot 10^3$	38	$3.819 \cdot 10^3$	$2.987 \cdot 10^3$	58	$6.409 \cdot 10^3$
$1.349 \cdot 10^3$	19	$1.729 \cdot 10^3$	$2.379 \cdot 10^3$	39	$3.939 \cdot 10^3$	$3.009 \cdot 10^3$	59	$6.549 \cdot 10^3$
$1.41 \cdot 10^3$	20	$1.83 \cdot 10^3$	$2.42 \cdot 10^3$	40	$4.06 \cdot 10^3$	$3.03 \cdot 10^3$	60	$6.69 \cdot 10^3$

2

$$3b) = B-K = \frac{K^2+K}{P} + \frac{P+1}{2}$$

2 (

$\frac{6+2}{2} = 4$

$6+1+2$

The Case P odd — Straight forward with no 1

Set $K = NP$ where N is any positive integer

$P = 1, 3, 5, 7, \ldots$ — since $P=1$ is allowed

$N = 1, 2, 3, 4, 5, \ldots$

There is no loss of generality K can be $1, 2, 3, \ldots$

$$B - NP = \frac{N^2P^2+NP}{P} + \frac{P+1}{2}$$

$\exists_R B$ for any N (may)

$$B = NP + N^2P + N + \frac{P+1}{2}$$

$$3c) \quad B = \frac{2(N^2P+NP+N) + P+1}{2}$$

$\frac{\text{even} + \text{even}}{2}$ OK

All odd P and any N gives integer B

---

difficult → The Case P even — Use 3c) $\frac{\text{even}+\text{odd}}{2}$ [illegible]

Set $K = N(P+1)$

$P+1 = 3, 5, 7, 9, \ldots$

$N = 1, 2, 3, \ldots$

K = all odd + 6, 10, 14, 18

excluded K's 2, 4, 8, 12, 16

$$B - N(P+1) = \frac{N^2(P+1)^2 + N(P+1)}{P} + \frac{P+1}{2}$$

$$N^2(P^2+2P+1) + NP + N$$

$$\frac{N^2P^2 + 2N^2P + NP + N^2 + N}{P} + \frac{P+1}{2}$$

$P = 2, 4, 6, 8, 10$

$N = 1, 2, 3, 4, 5$

K = only evens

loss of generality

$\therefore$ Use $K = N(P+1)$

$$2P[B - N(P+1)] = 2[N^2(P+1)^2 + N(P+1)] + P(P+1)$$

$$= (P+1)\{2[N^2(P+1)+N] + P\}$$

$B - N =$

$$2PB - 2PN(P+1) =$$

$$\frac{2PB}{P+1} - 2PN = 2[N^2(P+1)+N] + P$$

$$\frac{PB}{P+1} - PN = N^2P + N^2 + N + \frac{P}{2}$$

$H = \sqrt{1+16q}$ square

$K^2 + \ldots - 4q = 0$ $\quad K = \dfrac{-1 \pm \sqrt{1+16q}}{2}$

q	H^2	H		K	
0	1	1			
1	17				
2	33				
→ 3	49	7		3	
4	65		2		
→ 5	81	9		4	
…			6		Δ 3
14	225	15		7	
			2		
18	289	17		8	
			6		Δ 3
33	529	23		11	
			2		
39	625	25		12	
					Δ 3
60	961	31		15	
68	1089	33		16	
95	1521	39		19	
105	1681	41		20	

P even

K for which ∃ B

$B = \dfrac{K^2+K}{P} + K + \dfrac{P+1}{2}$

P even 2 none

4

But WANT $\dfrac{K^2+K}{P} = \dfrac{N}{2}$ $\quad$ N odd int, P even

$2K^2 + 2K - PN = 0$

$\dfrac{-2 \pm \sqrt{4 + 8PN}}{2} = -1 + \sqrt{1 + 2PN}$

$1 + 2PN =$ square $\quad$ P even

PN = W

sq = 1 + 2W = square

W = 4

P N

2

		PN = W	sq	sq^2
		4	3	9
4·2	8	12	5	25
3	12	24	7	49
4	16	40	9	81
5	20	60	11	121
6	24	84	13	169
7	28	112	15	225

80

169

168

PN =

Δ = 4·i

W = 4 + Δ

$\dfrac{W}{4} =$ 1, 3, 6, 10, 15, 21, 28

And this behavior is dangerous in a place like Israel and Palestine. You have millions of Christians fixated on Armageddon theology. They spend a great deal of time watching TV preachers, picking apart Bible verses, looking at headlines in the news, patching together pieces of information to create a sort of image well-intentioned Christians who actively oppose any kind of reconciliation among Israelis and Palestinians

$$B = \frac{[N^2(P+1)^2 + N(P+1)]}{P} + \frac{(P+1)}{2} + N(P+1)$$

$$B = \frac{2[\quad] + P(P+1) + 2NP(P+1)}{2P}$$

$$B = \frac{(P+1)}{2P}\{2N^2(P+1) + 2N + P + 2NP\}$$

even

$$B = \frac{\text{odd} \cdot \text{even}}{2\ \text{even}}$$

$$B = K + \frac{K^2+K}{P} + \frac{P+1}{2}$$

$P = 3$

$$K + \frac{K(K+1)}{2} + \frac{3}{2} \qquad \frac{3}{2} \text{ left over}$$

$$P = 4 \quad K + \frac{K(K+1)}{4} + \frac{5}{2} \qquad \text{difficulties}$$

$$\frac{1}{2}\left(\frac{\text{even}}{2} + \frac{5}{2}\right)$$

$$3c)\ B = \frac{2(N^2P + NP + N) + (P+1)}{2}$$

if P is odd, 1, 3, 5.
No loss of generality

if P is even
excluded K's

$$\frac{2N(PN+1) + P+1}{2}$$

So consider

$N^2P + NP + N$ must be odd

$K = 4q$
could also do
$q = 1, 2, \ldots$

$$B - K = \frac{K(K+1)}{P} + \frac{P+1}{2} \qquad \frac{2(N^2+N) + P+1}{2}$$

odd $NP(NP+1)$ no go

$$\frac{4q(4q+1)N}{4q}$$

$N^2P + N$ ok

but ∃ a loss of generality with $K = NP$

$NP = 4q$

excluded K's are 2, 4, 8, 12, 16

$$P = \frac{4q}{N} \qquad N(4q+1) + \frac{P+1}{2} \qquad \frac{2N(4q+1) + (P+1)}{2}$$

$P = 3$

$B = \frac{2(N^2P + NP + N) + P + 1}{2}$

$N^2 3 + N3 + N \quad +2$

$P=3$	N	B	$K = NP$	$J = K+P$
	1	9	3	6
	2	22	6	9
	3	41	9	12

$K = NP$

33 ✓

8, 7, 6, 5, 4, 3

16 × 9 = 144; 16 × 5 = 80; 16 × 7 = 112, 113; 16 × 8 = 128, 129

9, 21, 20, 19, 18, 17, 16, 15, 14, 13 — 153 ✓ (223)

6, 23, 24, 25, 26, 27, 28 — 153 ✓

161

$1 + 16q$ = square

$\sqrt{}$ $q = 3$ 49 $\frac{7-1}{2} = \frac{6}{2}$ 153

$\sqrt{}$ $q = 5$ 81 $9-1$ $K=3$

$q = 14$ 225

$K = \frac{-1 \pm \sqrt{1+16q}}{2}$

K an integer

$1 + 16q$ a square

$q = 0$

$\frac{K(K+1)}{2X}$ must be divisible by 2

i.e. $\frac{K(K+1)}{4X}$ must be an integer

$K = \ldots$

$X = \ldots$

$m = \ldots$

$K(K+1) = 4Xm$

$K(K+1) = 49$

$K^2 + K - 49 = 0$

A B. for all odd P. 1, 3, 5 …
and any N = 1, 2, 3, …

$\frac{2(K^2 + K) + P(P+1)}{2P}$

$B - K = \frac{2K(K+1) + P(P+1)}{2P}$

P even

$\frac{K(K+1)}{P}$ must be divisible by 2

$P = 2X \quad X = 1 \ldots$

$\frac{K^2+K}{2X} + \frac{2X+1}{2} =$ integer

$\frac{K^2+K}{2X} + X + \frac{1}{2} \qquad X = \frac{P}{2}$

$\frac{1}{2}\left(X + \frac{K^2+K}{X}\right)$

$K = mX$

$H = m^2X + m + 1$ even

$\frac{1}{2}\left(1 + m^2X + m\right)$

for what m's X's

X	m	H
		3
1	1	4
2	1	11
3	1	
4	1	

Page 5

The investigation centers on the relation of P to K

$P \geq K$

$P = (J - K)$ $\quad$ ~~$B - K = \frac{K+1}{2} + \frac{K+1}{2}$~~ $\neq K$

$2K = J$ $\quad$ $2K+1$

$B - K = \frac{K^2+K}{P} + \frac{P+1}{2}$

$B - P = \frac{P^2+P}{P} + \frac{P+1}{2}$

$B - P = (P+1)\frac{3}{2}$ $\quad$ P odd

OK!

$P = K+1$

$B - K = P - K + \frac{P+1}{2}$

$B - (P-1) = (P-1) + \frac{P+1}{2}$

$B = 2(P-1) + \frac{P+1}{2}$

$J = K+4$

$8B = K^2+K + (K+4)^2 + K+4$

$= K^2+K+K^2+8K+16+K+4$

$B = \frac{2K^2+10K+20}{8}$

$\frac{K^2+5K+10}{4}$ $\quad$ K even possible, K odd possible

---

$2B(J-K) = K^2+K+J^2+J$

$J = K$ no go

$P = 1$ $\quad$ $J = K+1$

$2B = K^2+K+(K+1)^2+K+1$

$= K^2+K+K^2+2K+1+K+1$

$= 2K^2+4K+2$ $\quad$ all

$B = K^2+2K+1 = (K+1)^2$ $\quad$ OK

$P = 2$ $\quad$ $J = K+2$

$2B = K^2+K+(K+2)^2+K+2$

$= K^2+K+K^2+4K+4+K+2$

$4B$ ~~$2B$~~ $= 2K^2+6K+6$

$B = \frac{K^2+3K+3}{2} = \frac{K(K+3)+3}{2}$ $\quad$ no go

none

$J = K+3$

$6B = K^2+K+(K+3)^2+K+3$

$= K^2+K+K^2+6K+9+K+3$

$= 2K^2+8K+12$

$B = \frac{2K^2+8K+12}{6}$

$B = \frac{K^2+4K+6}{3}$ $\quad$ $\frac{\text{even}}{3}$

K even, K odd $\quad$ $\frac{\text{odd}}{3}$

Possible

$J = K+5$

$10B = K^2+K + (K+5)^2 + K+5$

$= K^2+K+K^2+10K+25+K+5$

$10B = 2K^2+12K+30$

$B = \dfrac{K^2+6K+15}{5}$ possible

$J = K+6$ no way

$12B = K^2+K+(K+6)^2+K+6$

$= K^2+K+K^2+12K+36+K+6$ K even no

$B = \dfrac{2K^2+14K+42}{12} = \dfrac{K^2+7K+21}{6}$ K odd no

$J = K$ no

$J = K+1$ all values k

$J = K+2$ none

$J = K+3$ possible

$J = K+4$ possible

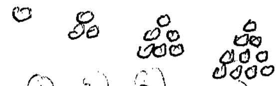
$J = K+5$ possible

$J = K+6$ none

SEQUENCES

04-02-27

TRIANGULAR NUMBERS



	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫
TN:	1	3	6	10	15	21	28	36	45	55	66	78
Δ(TN)		2	3	4	5	6	7	8	9			
P = Pentahedral = Σ(TN)	1	4	10	20	35	56	84	120	165	220	286	364
SQUARES □	1	4	9	16	25	36	49	64	81	100	121	144
Δ□		3	5	7	9	11	13	15	17	19	21	23

$\frac{T_{n+1}}{T_n} \to \frac{117}{99} = 1.\overline{18}$

$\frac{P_{n+1}}{P_n} \to 1.\overline{27}$

$T_n + T_{n+1} = (n+1)^2 = \square_{n+1}$

$T_{n+1} = T_n + n$ Recursion

$\square_n = n^2$

$\Delta(TH) = TN$

$\Sigma(TN) = TH$

$\Delta(TN) = N$

$T_n = an^2 + bn + c$

$a = \frac{1}{2},\ b = \frac{1}{2},\ c = 0$

$(TN)_{n+1} - (TN)_n = n+1$

$(TN)_{n+1} + (TN)_n = (n+1)^2$

P_n recursion?

$P_{n+1} = P_n + T_n = P_n + \frac{n(n+1)}{2}$ ✓

$(TN)_{n+1} = \frac{(n+1) + (n+1)^2}{2}$ EXPLICIT

EXPLICIT $(TN)_n = \frac{n+n^2}{2} = \frac{n(n+1)}{2} = \sum_1^n i$

$P_n = \sum_1^n T_i = \sum_1^n \frac{i(i+1)}{2} = \sum_m^n \sum_1^m i\ ?$

$P_n = an^3 + bn^2 + cn + d$

$1 = a + b + c + d$

$4 = 8a + 4b + 2c + d$

$10 = 27a + 9b + 3c + d$

$20 = 64a + 16b + 4c + d$

$> 3 = 7a + 3b + c$

$> 6 = 19a + 5b + c$

$> 10 = 37a + 7b + c$

$> 3 = 12a + 2b$

$> 4 = 18a + 2b$

$> 1 = 6a$

$a = \frac{1}{6},\ b = \frac{1}{2},\ c = \frac{1}{3};\ d = 0$

$P_n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n = \frac{n(n+1)(n+2)}{6} = P_n$ ✓ EXPLICIT

$P_{12} = 364$ ✓

Generalized Summations

$\frac{(n+1)!}{(n-1)!2!}$ means $\frac{(n+1)n}{2}$; $\frac{(n+2)!}{(n-1)!\,3!}$

$T_n = \frac{(n+1)!}{(n-1)!2!}$

$P_n = \frac{(n+2)!}{(n-1)!\,3!}$

FIBONACCI

Fibonacci Numbers

The recursion formula for the Fibonacci numbers is $F_{r+1} = F_r + F_{r-1}$
Beginning with the initial numbers 1 and 1 the recursion formula gives the Fibonacci sequence:

1,1,2,3,5,8,13,21,34,55,89,144,233,.....

The limit of the ratio between two successive numbers, $\lim(F_{r+1}/F_r)$ as r increases is $(1+\sqrt{5})/2$
[It is to be noted that whatever the initial pair of numbers, the ratio limit is always $(1+\sqrt{5})/2$]

The Divine Proportion or Golden Mean

The Divine Proportion is A:B :: B:A+B
Dividing by B and letting A/B = x, we have $x = 1/(1+x)$ or $x^2 + x - 1 = 0$
The solutions to this quadratic equation are $x = (1 \pm \sqrt{5})/2$
By convention the positive root, $x = (1 + \sqrt{5})/2$, is designated by Φ
This value is called the Golden Mean or Divine Proportion
[Here we shall designate the negative root $x = (1 - \sqrt{5})/2$ by φ]

Explicit Formula

If we wish to know the value of the 110th Fibonacci number, for example, and do not want to repeatedly apply the recursion formula, we need an **explicit** formula which gives the value of F_n when we are given only n. While it is not always possible to derive an explicit formula from a recursion formula, in the case of sequences like the Fibonacci sequence it is. The explicit formula for Fibonacci numbers is:

$$F_n = (\Phi^n - \varphi^n)/\sqrt{5}$$

The above is a brief introduction to the arithmetic properties of the golden mean. There are also many geometric and esthetic properties and many manifestations in nature. [An example, the loops in the analemma. The northern loop is to the southern loop as the southern loop is to the whole year. This is roughly true at present but the shape of the analemma evolves over thousands of years.]

For more information on the mathematical, esthetic, and historical aspects of Φ, I recommend

The Divine Proportion by H.E.Huntley Dover Publications 1970
and Vol XVI no 4 Winter 1991 of PARABOLA magazine.

FIBRAM1.WPD February 13, 2004

Fibonacci Numbers

The recursion formula for the Fibonacci numbers is $F_{r+1} = F_r + F_{r-1}$
Beginning with the initial numbers 1 and 1 the recursion formula gives the Fibonacci sequence:
1,1,2,3,5,8,13,21,34,55,89,144,233,.....
The limit of the ratio between two successive numbers, $\lim(F_{r+1}/F_r)$ as r increases is $(1+\sqrt{5})/2$
[It is to be noted that whatever the initial pair of numbers, the ratio limit is always $(1+\sqrt{5})/2$]

The Divine Proportion or Golden Mean

The Divine Proportion is A:B :: B:A+B
Dividing by B and letting A/B = x, we have $x = 1/(1+x)$ or $x^2 + x - 1 = 0$
The solutions to this quadratic equation are $x = (1 \pm \sqrt{5})/2$
By convention the positive root, $x = (1 + \sqrt{5})/2$, is designated by Φ
This value is called the Golden Mean or Divine Proportion
[Here we shall designate the negative root $x = (1 - \sqrt{5})/2$ by ϕ]

Explicit Formula

$\varphi = \frac{\sqrt{5}-1}{2}$

If we wish to know the value of the 110th Fibonacci number, for example, and do not want to repeatedly apply the recursion formula, we need an **explicit** formula which gives the value of F_n when we are given only n. While it is not always possible to derive an explicit formula from a recursion formula, in the case of sequences like the Fibonacci sequence it is. The explicit formula for Fibonacci numbers is:

$$F_n = (\Phi^n - \phi^n)/\sqrt{5}$$

The above is a brief introduction to the arithmetic properties of the golden mean. There are also many geometric and esthetic properties and many manifestations in nature. [An example, the loops in the analemma. The northern loop is to the southern loop as the southern loop is to the whole year. This is roughly true at present but the shape of the analemma evolves over thousands of years.] The ratio of $\frac{\text{Common}}{\text{Private}} = \frac{\text{Private}}{\text{Common+Private}}$ i.e. P = 62% of Whole for synchronic, C = 38% of Whole, or vice versa for diachronic

For more information on the mathematical, esthetic, and historical aspects of Φ, see

The Divine Proportion by H.E.Huntley Dover Publications 1970 V - E - 3
Mathematics Appreciation –Theoni Pappas I - A - I
Math and the Mona Lisa –Bulent Atalay V - B - 2
Science and the Future Year Book 1977
Number Theory: The Fibonacci Sequence –Verner E. Hagget Jr. p 178
PARABOLA, Vol XVI no 4, Winter 1991

DIVPROP.WPD June 2, 2006

THE DIVINE PROPORTION

The Divine Proportion or Golden Section: A : B :: B : A+B

Set A = 1 and B = x, the proportion becomes: $x^2 - x - 1 = 0$
This quadratic equation has two solutions: $x = (1+\sqrt{5})/2$ and $x = (1-\sqrt{5})/2$
The quantity, $(1+\sqrt{5})/2$, is customarily designated by Φ and stands for the Golden Section
The negative of the second solution, $(\sqrt{5}-1)/2$, is usually designated by ϕ.
Numerically, $\Phi = 1.6180338887....$ and $\phi = 0.6180338887......$ $\Phi = 1 + \phi$ and $\Phi = 1/\phi$

The Divine Proportion is mathematically related to the Fibonacci Sequence,
F[1,1] = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597 ......
Which is generated by the recursion formula, $F_{n+2} = F_{n+1} + F_n$
One connection of the Fibonacci sequence to Φ is through the ratios of successive terms.
The lim as $n \to \infty$ of $F_{n+1} / F_n = \Phi$ and the limiting ratio for $F_{n+2}/F_n = \Phi^2$, for $F_{n+3}/F_n = \Phi^3$, etc

Other relations between Φ and the Fibonacci sequence involve powers of Φ:

For odd exponents:	For even exponents
$\Phi - 1/\Phi = 1$	$\Phi^2 + 1/\Phi^2 = 3$
$\Phi^3 - 1/\Phi^3 = 4$	$\Phi^4 + 1/\Phi^4 = 7$
$\Phi^5 - 1/\Phi^5 = 11$	$\Phi^6 + 1/\Phi^6 = 18$
$\Phi^7 - 1/\Phi^7 = 29$	$\Phi^8 + 1/\Phi^8 = 47$
$\Phi^9 - 1/\Phi^9 = 76$	$\Phi^{10} + 1/\Phi^{10} = 123$
$\Phi^{11} - 1/\Phi^{11} = 199$	$\Phi^{12} + 1/\Phi^{12} = 322$

Both the odd exponent sequence: A[1,4] = 1, 4, 11, 29, 76, 199, 521, 1364 ....
And the even exponent sequence: A[3,7] = 3, 7, 18, 47, 123, 322, 843, 2207 ....
follow the alternate term recursion formula: $A_{n+2} = 3\,A_{n+1} - A_n$

If the two sequences A[1,4} and A[3,7] are combined maintaining numerical order, we obtain:
L[1.3] = 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207 ....
which is known as the Lucas sequence. The Lucas sequence follows the same recursion formula, $F_{n+2} = F_{n+1} + F_n$, as the Fibonacci sequence. For both sequences the lim as $n \to \infty$ of $F_{n+1} / F_n = \Phi$, lim $F_{n+2}/F_n = \Phi^2$, lim $F_{n+3}/F_n = \Phi^3$, etc

If the F[1,1] sequence is partitioned into two sequences built with alternating terms, viz,
A[1,2] = 1, 2, 5, 13, 34, 89, 133, 610, 1597 ....
and A[1,3] = 1, 3, 8, 21, 55, 144, 377, 987, 2584....
These sequences as well as A[1,3] and A[3.7] follow the alternate term recursion formula:

$$A_{n+2} = 3\,A_{n+1} - A_n$$

Summarizing:
The complete sequences follow the recursion formula, $F_{n+2} = F_{n+1} + F_n$,
The alternate term sequences follow the recursion formula, $A_{n+2} = 3\,A_{n+1} - A_n$

DIVPROP2.WPD June 5, 2006

ON SEQUENCES

Recursion formulae:

F[a,b] $F_{n+2} = F_{n+1} + F_n$ [EVERY TERM]

A[a,b] $A_{n+2} = 3\,A_{n+1} - A_n$ [EVERY OTHER TERM]

B[a,b] $B_{n+2} = 4\,B_{n+1} + B_n$ [EVERY THIRD TERM]

C[a,b] $C_{n+2} = 7\,C_{n+1} - C_n$ [EVERY FOURTH TERM]

D[a,b] $D_{n+2} = 11D_{n+1} + D_n$ [EVERY FIFTH TERM]

Note:
The coefficients of the n+1 terms are members of the Lucas sequence

---

$\Sigma_{n+3} = 2\,\Sigma_{n+2} - \Sigma_n$ Recursion formula for the summation sequences

---

Explicit formulae: F[1,1] is the Fibonacci sequence:

F[1,1] = 1,1,2,3,5,8,13,21,34,55,89,144,... $F_n = (\Phi^n - \phi^n)/\sqrt{5}$

Σ[1,1] = 1,2,4,7,12,20,33,54,88,143,... $\Sigma_n = (P\Phi^n - Q\phi^n)/\sqrt{5} - 1$

A[1,3] = 1,3,8,21,55,144,377,987,2584,... $A_n = (P^n - Q^n)/\sqrt{5}$

A[1,2] = 1,2,5,13,34,89,233,610,1597,... $A_n = (\Phi Q^n - \phi P^n)/\sqrt{5}$

L[1,3] is the Lucas sequence

L[1,3] = 1,3,4,7,11,18,29,47,76,123,199,.322 $L_n = \Phi^n + \phi^n$

Σ[1,3] = 1,4,8,15,26,44,73,120,196,.... $\Sigma_n = P\Phi^n + Q\phi^n - 3$

A[3,7] = 3,7,18,47,123,322,843,2207,5778... $A_n = P^n + Q^n$

A[1,4] = 1,4,11,29,76,199,521,1364,3571... $A_n = P^n\Phi + Q^n\phi$

$\Phi = (1 + \sqrt{5})/2, \quad -\phi = (1 - \sqrt{5})/2$ $P = (3 + \sqrt{5})/2, \quad Q = (3 - \sqrt{5})/2$

$\Phi + \phi = +1, \quad \Phi - \phi = \sqrt{5}, \quad \Phi \cdot \phi = -1$ $P + Q = +3, \quad P - Q = \sqrt{5}, \quad P \cdot Q = +1$

$\Phi^2 = \phi^{-2} = P = Q^{-1}$ $Q = P^{-1} = \phi^2 = \Phi^{-2}$

$P^n\Phi = \Phi^{2n+1} \qquad P\Phi^n = \Phi^{n+2}$ $Q^n\phi = \phi^{2n+1} \qquad Q\phi^n = \phi^{n+2}$

FIBSEQ.WPD June 15, 2006

THE FIBONACCI and LUCAS SEQUENCES

The Fibonacci numbers are a sequence of numbers based on the recursion formula,

1) $F_{n+2} = F_{n+1} + F_n$

The initial numbers for the sequence are 1 and 1, leading to the sequence,

F[1,1] = 1, 1 ,2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584,....

18

A second important sequence based on the same recursion formula but beginning with 1 and 3 is:

L[1,3] = 1, 3, 4. 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571,...

This sequence is known as the Lucas sequence. $= \Phi^n + \varphi^n$ 17 $n \uparrow \frac{L_n}{F_n} \to \sqrt{5}$ $\frac{17}{17}$

Whatever the initial numbers are, 1 and 1, 1 and 3, or any n_1 and n_2,
the limiting value of the ratio between two successive numbers, F_{n+1}/F_n, or L_{n+1}/L_n $\frac{17}{18} \to 2 - \varphi$
as n increases is always = $(1+\sqrt{5})/2$ $= \frac{5-\sqrt{5}}{2}$

This quantity, $(1+\sqrt{5})/2$ = 1.61803398874989...., is usually symbolized with Φ and is called
THE GOLDEN SECTION or DIVINE PROPORTION

In addition to defining sequences by <u>recursion</u> equations, such as 1) $F_{n+2} = F_{n+1} + F_n$
it is also possible to define sequences by <u>explicit</u> equations in which the value of F_n is given directly as a function of n. The explicit formula for the Fibonacci sequence, F[1,1], is

2) $F_n = (\Phi^n - \phi^n)/\sqrt{5}$ where $\phi = (1 - \sqrt{5})/2$

And the explicit formula for the Lucas sequence L[1,3] is

3) $L_n = \Phi^n + \phi^n$

Also of interest are the sequences giving the term by term sums of the above sequences.
The recursion formula for the summation sequences is 4) $S_{n+3} = 2\, S_{n+2} - S_n$

For the Fibonacci sequence,
S[1,1] = 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986, 1596,....; $S_n = F_{n+2} - 1$
The explicit formula for this sequence is: 5) $S_n = (\Phi^{n+2} - \phi^{n+2})/5 - 1$

For the Lucas sequence,
S[1,3] = 1, 4, 8, 15, 26, 44, 73, 120, 196, 319, 518, 840, 1361, 2204,.....; $S_n = L_{n+2} - 3$
with explicit formula 6) $S_n = \Phi^{n+2} + \phi^{n+2} - 3$

RECFORM.WPC June 18, 2006

RECURSION FORMULAE RELATED TO THE FIBONACCI AND LUCAS SEQUENCES

The following notations will be used:

In refers to the natural numbers:
1,2,3,4,5,6,7,8,9,10.11.12.13.14.15.16.17.18.19.20.21....

Fn refers to the Fibonacci sequence:
1,1 ,2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584,....

Ln refers to the Lucas sequence:
1, 3, 4. 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571,...

The customary recursion formula for Fn or Ln is: $F_{n+1} = F_n + F_{n-1}$
This is but one of many "symmetric" recursion formulae each valid for both Fn and Ln. Others include:

$$F_{n+1} = 1 \cdot F_n + F_{n-1}$$
$$F_{n+2} = 3 \cdot F_n - F_{n-2}$$
$$F_{n+3} = 4 \cdot F_n + F_{n-3}$$
$$F_{n+4} = 7 \cdot F_n - F_{n-4}$$
$$F_{n+5} = 11 \cdot F_n + F_{n-5}$$
$$F_{n+6} = 18 \cdot F_n - F_{n-6}$$

Note that the coefficients of F_n are members of the Ln sequence. This fact allows the general formula for the sequence of symmetric recursion formulae to be written as:

$$F_{n+r} = L_r \cdot F_n + (-1)^{r+1} \cdot F_{n-r}$$

where n designates the nth Fibonacci term and r = 1,2,3,4,5,.... And where L_r takes on the r[th] value of the Ln sequence. Since $L_r = \Phi^r + \phi^r$, the general formula may also be written as: $F_{n+r} = (\Phi^r + \phi^r) \cdot F_n + (-1)^{r+1} \cdot F_{n-r}$

Other interesting recursion formulae interrelate the Fn and Ln sequences:

$1 \cdot L_n = F_{n+1} + F_{n-1}$	$5 \cdot F_n = L_{n+1} + L_{n-1}$
$1 \cdot L_n = F_{n+2} - F_{n-2}$	$5 \cdot F_n = L_{n+2} - L_{n-2}$
$2 \cdot L_n = F_{n+3} + F_{n-3}$	$10 \cdot F_n = L_{n+3} + L_{n-3}$
$3 \cdot L_n = F_{n+4} - F_{n-4}$	$15 \cdot F_n = L_{n+4} - L_{n-4}$
$5 \cdot L_n = F_{n+5} + F_{n-5}$	$25 \cdot F_n = L_{n+5} + L_{n-5}$
----------------------	---------------------------
$Fr \cdot L_n = F_{n+r} + (-1)^{r+1} \cdot F_{n-r}$	$5 \cdot Fr \cdot F_n = L_{n+r} + (-1)^{r+1} \cdot L_{n-r}$

The richness of the interrelations between these sequences may be one reason they occur so often in nature. In fact, such sequences may be nature's natural numbers, rather than the sequence of integers that are basic to human cultures. In contrast, integers appear to have only one recursion formula: $I_{n+1} = 2 \cdot I_n - I_{n-1}$

FIBONF.MCD — FIBONACCI NUMBERS F[1,1] AND F[1,3] — June 6, 2006

Explicit Formulae

$$\Phi := \frac{\left(1+\sqrt{5}\right)}{2} \qquad \phi := \frac{\left(1-\sqrt{5}\right)}{2}$$

$$F(n) := \frac{\left(\Phi^{n}-\phi^{n}\right)}{\sqrt{5}} \qquad G(n) := \Phi^{n}+\phi^{n}$$ = Lucas Sequence

$n := 1, 2 .. 15$

$F(n) =$	$F(n+14) =$	$n =$	$n+14 =$	$G(n) =$	$G(n+14) =$
1	610	1	15	1	1364
1	987	2	16	3	2207
2	1597	3	17	4	3571
3	2584	4	18	7	5778
5	4181	5	19	11	9349
8	6765	6	20	18	15127
13	10946	7	21	29	24476
21	17711	8	22	47	39603
34	28657	9	23	76	64079
55	46368	10	24	123	103682
89	75025	11	25	199	167761
144	121393	12	26	322	271443
233	196418	13	27	521	439204
377	317811	14	28	843	710647
610	514229	15	29	1364	1149851

Explicit Formulae

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$ Recursion Formula

$f(n) := \frac{\Phi^n}{\sqrt{5}}$ $F(n) := \frac{(\Phi^n - \phi^n)}{\sqrt{5}}$ F(n) = F(n-1) + F(n-2)

$n := 1, 2 .. 17$

n =	f(n) =	round(f(n)) =	F(n) =
1	0.7236068	1	1
2	1.1708204	1	1
3	1.8944272	2	2
4	3.0652476	3	3
5	4.9596748	5	5
6	8.0249224	8	8
7	12.9845971	13	13
8	21.0095195	21	21
9	33.9941166	34	34
10	55.0036361	55	55
11	88.9977528	89	89
12	144.0013889	144	144
13	232.9991416	233	233
14	377.0005305	377	377
15	609.9996721	610	610
16	987.0002026	987	987
17	1596.9998748	1597	1597

$\Phi = 1.618034$ $\Phi + \phi = 1$

$\phi = -0.618034$ note minus sign

$\sqrt{5} = 2.236068$

FIBON13.MCD

LUCAS NUMBERS

L[(1,3]

June 5, 2006

$$\Phi := \frac{\left(1+\sqrt{5}\right)}{2} \qquad \phi := \frac{\left(1-\sqrt{5}\right)}{2}$$

$$\sqrt{5} = 2.23606798 \qquad \Phi - \phi = 2.23606798 \qquad \Phi + \phi = 1 \qquad \Phi \cdot \phi = -1$$

$$H(n) := \frac{\left[\left(\Phi^{n+1} + \frac{\Phi^n}{\Phi}\right) - \left(\phi^{n+1} + \frac{\phi^n}{\phi}\right)\right]}{\sqrt{5}} \qquad G(n) := \frac{\left[\left(\Phi^{n+1} + \Phi^{n-1}\right) - \left(\phi^{n+1} + \phi^{n-1}\right)\right]}{\sqrt{5}}$$

$$n := 1, 2 .. 15 \qquad J(n) := \frac{\left(\Phi^n + \phi^n\right)\cdot(\Phi - \phi)}{\sqrt{5}} \qquad L(n) := \Phi^n + \phi^n$$

n =	H(n) =	G(n) =	J(n) =	L(n) =
1	1	1	1	1
2	3	3	3	3
3	4	4	4	4
4	7	7	7	7
5	11	11	11	11
6	18	18	18	18
7	29	29	29	29
8	47	47	47	47
9	76	76	76	76
10	123	123	123	123
11	199	199	199	199
12	322	322	322	322
13	521	521	521	521
14	843	843	843	843
15	1364	1364	1364	1364

FIBONSUM

June 14, 2006

SUMMATION SEQUENCES
$\Sigma[1,1]$ and $\Sigma[1,3]$

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$ $P := \frac{(3+\sqrt{5})}{2}$ $Q := \frac{(3-\sqrt{5})}{2}$

The recursion formula for the summation sequences is S(n+3) = 2 S(n+2) - S(n)

$\Sigma[1,3]$ $V(n) := \phi^{n+2} + \Phi^{n+2} - 3$ $\Sigma[1,1]$ $W(n) := -1 + \frac{(\Phi^{n+2} - \phi^{n+2})}{\sqrt{5}}$

$Z(n) := Q\cdot\phi^{n} + P\cdot\Phi^{n} - 3$ $Y(n) := -1 + \frac{(P\cdot\Phi^{n} - Q\cdot\phi^{n})}{\sqrt{5}}$

$n := 1, 2 .. 15$

ΣF_n

n =	V(n) =	W(n) =	Z(n) =	Y(n) =
1	1	1	1	1
2	4	2	4	2
3	8	4	8	4
4	15	7	15	7
5	26	12	26	12
6	44	20	44	20
7	73	33	73	33
8	120	54	120	54
9	196	88	196	88
10	319	143	319	143
11	518	232	518	232
12	840	376	840	376
13	1361	609	1361	609
14	2204	986	2204	986
15	3568	1596	3568	1596

A[1,2]

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\quad \phi := \frac{(1-\sqrt{5})}{2}$ $\quad P := \frac{(3+\sqrt{5})}{2}$ $\quad Q := \frac{(3-\sqrt{5})}{2}$

$A(n) := \frac{(\Phi \cdot Q^n - \phi \cdot P^n)}{\sqrt{5}}$ $\quad B(n) := \frac{(\Phi \cdot \phi^{2n} - \phi \cdot \Phi^{2n})}{\sqrt{5}}$

$n := 1, 2 .. 12$

n =	A(n) =	B(n) =
1	1	1
2	2	2
3	5	5
4	13	13
5	34	34
6	89	89
7	233	233
8	610	610
9	1597	1597
10	4181	4181
11	10946	10946
12	28657	28657

FIBONA13.MCD | ALTERNATE FIBONACCI NUMBERS A[1,3] | June 2, 2006

Explicit Formulae

$P := \frac{(3+\sqrt{5})}{2}$ $Q := \frac{(3-\sqrt{5})}{2}$ $\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$

Φ^2 = 1 + Φ = P ϕ^2 = 1 + ϕ = Q

$A(n) := \frac{(P^n - Q^n)}{\sqrt{5}}$ $B(n) := \frac{\left[(1+\Phi)^n - (1+\phi)^n\right]}{\sqrt{5}}$ $C(n) := \frac{(\Phi^{2n} - \phi^{2n})}{\sqrt{5}}$

n := 1, 2 .. 12

n =	A(n) =	B(n) =	C(n) =
1	1	1	1
2	3	3	3
3	8	8	8
4	21	21	21
5	55	55	55
6	144	144	144
7	377	377	377
8	987	987	987
9	2584	2584	2584
10	6765	6765	6765
11	17711	17711	17711
12	46368	46368	46368

FIBONA14.MCD

~~FIBONACCI~~ LUCAS NUMBERS
A[1,4]

June 6, 2006

$\Phi := \frac{\left(1+\sqrt{5}\right)}{2}$ $\phi := \frac{\left(1-\sqrt{5}\right)}{2}$ $P := \frac{\left(3+\sqrt{5}\right)}{2}$ $Q := \frac{\left(3-\sqrt{5}\right)}{2}$

$R(n) := \Phi \cdot P^{n-1} + \phi \cdot Q^{n-1}$ $H(n) := \frac{\left[\left(P \cdot Q^{n-1} - Q \cdot P^{n-1}\right) + 4 \cdot \left(P^{n-1} - Q^{n-1}\right)\right]}{\sqrt{5}}$

$n := 1, 2 .. 15$

n =	H(n) =	R(n) =
1	1	1
2	4	4
3	11	11
4	29	29
5	76	76
6	199	199
7	521	521
8	1364	1364
9	3571	3571
10	9349	9349
11	24476	24476
12	64079	64079
13	167761	167761
14	439204	439204
15	1149851	1149851

LUCAS

FIBONACCI NUMBERS
A[3,7]

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$ $P := \frac{(3+\sqrt{5})}{2}$ $Q := \frac{(3-\sqrt{5})}{2}$

$M(n) := P^n + Q^n$ $H(n) := \frac{\left[3\cdot\left(P\cdot Q^{n-1} - Q\cdot P^{n-1}\right) + 7\cdot\left(P^{n-1} - Q^{n-1}\right)\right]}{\sqrt{5}}$

$n := 1, 2 .. 15$

n =	M(n) =	H(n) =
1	3	3
2	7	7
3	18	18
4	47	47
5	123	123
6	322	322
7	843	843
8	2207	2207
9	5778	5778
10	15127	15127
11	39603	39603
12	103682	103682
13	271443	271443
14	710647	710647
15	1860498	1860498

FIBONex1.MCD | FIBONACCI NUMBERS F(n) | January 7, 2004

Explicit Formulae | $n := 1, 2 .. 20$ | Recursion Formula

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\qquad \phi := \frac{(1-\sqrt{5})}{2}$

F(n) = F(n-1) + F(n-2)

$f(n) := \frac{\Phi^n}{\sqrt{5}}$ $\qquad g(n) := \frac{\Phi^{2n}}{\sqrt{5}}$ $\qquad h(n) := \frac{\Phi^{\frac{n}{2}}}{\sqrt{5}}$ $\qquad j(n) := \frac{\left(\frac{\Phi}{\phi}\right)^n}{\sqrt{5}}$

n =	round(f(n)) =	round(g(n)) =	round(h(n)) =	round(j(n)) =
1	1	1	1	-1
2	1	3	1	3
3	2	8	1	-8
4	3	21	1	21
5	5	55	1	-55
6	8	144	2	144
7	13	377	2	-377
8	21	987	3	987
9	34	2584	4	-2584
10	55	6765	5	6765
11	89	17711	6	-17711
12	144	46368	8	46368
13	233	121393	10	-121393
14	377	317811	13	317811
15	610	832040	17	-832040
16	987	$2.178309\cdot10^6$	21	$2.178309\cdot10^6$
17	1597	$5.702887\cdot10^6$	27	$-5.702887\cdot10^6$
18	2584	$1.493035\cdot10^7$	34	$1.493035\cdot10^7$
19	4181	$3.908817\cdot10^7$	43	$-3.908817\cdot10^7$
20	6765	$1.023342\cdot10^8$	55	$1.023342\cdot10^8$

$\Phi = 1.618034$

Powers skip
Roots insert
/ ϕ skips and oscillates

$\phi = -0.618034$ note minus sign

$\sqrt{5} = 2.236068$ $\qquad \Phi + \phi = 1$

$\frac{\Phi^{3n}}{\sqrt{5}}$ skips 2 as does $\frac{\left(\frac{\Phi}{\varphi^2}\right)^n}{\sqrt{5}}$

$(\varphi\Phi)^2 = 1$

	Nearest integer
	1 1
$\Phi = 1.618034$	2
$2\Phi = 3.236068$	3
$3\Phi = 4.854102$	5
$5 \cdot \Phi = 8.090170$	8
$8 \cdot \Phi = 12.944272$	13
$13 \cdot \Phi = 21.034442$	21
$21 \cdot \Phi = 33.978714$	34
$34 \cdot \Phi = 55.013156$	55
. . . .	

$$F_n \Phi \to F_{n+1} \quad \text{or} \quad \frac{F_{n+1}}{F_n} \to \Phi$$

JOURNYEAR01
TY1FIBON.LST

PRINT: \LP SETUPLP
\WS LWS.BAT
PRINT B:TY1FIBON.LST
ESC $\Phi = 1.6180339\ldots$

phi = 1.618034...

$\text{round}\left(\frac{\Phi^N}{\sqrt{5}}\right) = f_N$ to nearest integer

FIBONACCI NUMBERS

N	f_N	PHI^N		$PHI^N/SQRT(5)$
1	1	1.62	2	.72
2	1	2.62	3	1.17
3	2	4.24	4	1.89
4	3	6.85	7	3.07
5	5	11.09	11	4.96
6	8	17.94	18	8.02
7	13	29.03	29	12.98
8	21	46.98	47	21.01
9	34	76.01	76	33.99
10	55	122.99	123	55.00
11	89	199.01	199	89.00
12	144	322.00		144.00
13	233	521.00		233.00
14	377	843.00		377.00
15	610	1364.00		610.00
16	987	2207.00		987.00
17	1597	3571.00		1597.00
18	2584	5778.00		2584.00
19	4181	9349.00		4181.00
20	6765	15127.00		6765.00
21	10946	24476.00		10946.00
22	17711	39603.00		17711.00
23	28657	64079.00		28657.00
24	46368	103682.00		46368.00
25	75025	167761.00		75025.00
26	121393	271443.00		121393.00
27	196418	439204.00		196418.00
28	317811	710647.00		317811.00
29	514229	1149851.00		514229.00
30	832040	1860498.00		832040.00
31	1346269	3010349.00		1346269.00
32	2178309	4870847.00		2178309.00
33	3524578	7881196.00		3524578.00
34	5702887	12752043.00		5702887.00
35	9227465	20633239.00		9227465.00
36	14930352	33385282.00		14930352.00
37	24157817	54018521.00		24157817.00
38	39088169	87403803.00		39088169.00
39	63245986	141422324.00		63245986.00

↓ $\text{trunc}\left[\frac{\Phi^N}{\sqrt{5}}\right]$

$f_N = \left[\frac{\Phi^N}{\sqrt{5}}\right]$ nearest integer

Fibonacci (Golden)

$F(n+1) = F(n) + F(n-1)$ $F(1)=1, \quad F(0)=0$

$\varphi^2 - \varphi - 1 = 0$ $\varphi = 1.618034$

$1, 1, 2, 3, 5, 8, 13, 21, \cdots$

$\lim_{n \to \infty} \frac{F(n+1)}{F(n)} = \varphi$ $\frac{1 \pm \sqrt{5}}{2}$

Padovan (Plastic)
see Sci. Am. June 1996 p. 102

$P(n+1) = P(n-1) + P(n-2)$

$p^3 - p - 1 = 0$ $p = 1.324718$

$P(n+1) = P(n) + P(n-4)$

$1, 1, 1, 2, 3, 4, 5, 7, 9, 12, 16, 21, \cdots$

$\lim_{n \to \infty} \frac{P(n+1)}{P(n)} = p$?

$X^4 - X - 1 = 0$

$X^n - X - 1 = 0$ ~ ?

$X^0 - X - 1 = 0$ $X = 0$

$1+2=3$, $3+4=7$, $7+8=15$, $15+16=31$, $31+32=63$, $63+64=127$, $127+128=255$,
$255+256=511$, $511+512=1023$

FIBON1.MCD

FIBONACCI NUMBERS
F(n)
Explicit Formulae

January 4, 2004

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$

Recursion Formula

$f(n) := \frac{\Phi^n}{\sqrt{5}}$ $F(n) := \frac{(\Phi^n - \phi^n)}{\sqrt{5}}$

F(n) = F(n-1) + F(n-2)

$n := 1, 2 .. 20$

n =	f(n) =	round(f(n)) =	F(n) =
1	0.7236068	1	1
2	1.1708204	1	1
3	1.8944272	2	2
4	3.0652476	3	3
5	4.9596748	5	5
6	8.0249224	8	8
7	12.9845971	13	13
8	21.0095195	21	21
9	33.9941166	34	34
10	55.0036361	55	55
11	88.9977528	89	89
12	144.0013889	144	144
13	232.9991416	233	233
14	377.0005305	377	377
15	609.9996721	610	610
16	987.0002026	987	987
17	1596.9998748	1597	1597
18	2584.0000774	2584	2584
19	4180.9999522	4181	4181
20	6765.0000296	6765	6765

$\Phi = 1.618034$ $\Phi + \phi = 1$

$\phi = -0.618034$ note minus sign

$\sqrt{5} = 2.236068$

FIBON2.MCD | SUBSETS OF FIBONACCI NUMBERS | January 6, 2004

Explicit Formulae

Recursion Formula

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$

A(n) = 3 A(n-1) - A(n-2)

$\frac{\Phi^{2n}}{\sqrt{5}}$: $a(n) := \frac{(1+\Phi)^n}{\sqrt{5}}$ $A(n) := \frac{\left[(1+\Phi)^n - (1+\phi)^n\right]}{\sqrt{5}}$ or $A(n) := \frac{\left(\Phi^{2n} - \phi^{2n}\right)}{\sqrt{5}}$

$n := 1, 2 .. 15$

n =	a(n) =	round(a(n)) =	A(n) =
1	1.1708204	1	1
2	3.0652476	3	3
3	8.0249224	8	8
4	21.0095195	21	21
5	55.0036361	55	55
6	144.0013889	144	144
7	377.0005305	377	377
8	987.0002026	987	987
9	2584.0000774	2584	2584
10	6765.0000296	6765	6765
11	17711.0000113	17711	17711
12	46368.0000043	46368	46368
13	121393.0000016	121393	121393
14	317811.0000006	317811	317811
15	832040.0000002	832040	832040

$\Phi = 1.618034$ $\Phi + \phi = 1$

A(n) uses initial values 0 and 1

$\phi = -0.618034$ note minus sign

$\sqrt{5} = 2.236068$

FIBON3.MCD | SUBSETS OF FIBONACCI NUMBERS (a skip one subset) | January 7, 2004

Explicit Formulae

$\Phi := \frac{(1+\sqrt{5})}{2}$ $\phi := \frac{(1-\sqrt{5})}{2}$

Recursion Formula

$d(n) := \frac{\Phi^{2n+1}}{\sqrt{5}}$ $D(n) := \frac{(\Phi^{2n+1} - \phi^{2n+1})}{\sqrt{5}}$

D(n) = 3 D(n-1) - D(n-2)

$n := 1, 2 .. 15$

n =	d(n) =	round(d(n)) =	D(n) =
1	1.8944272	2	2
2	4.9596748	5	5
3	12.9845971	13	13
4	33.9941166	34	34
5	88.9977528	89	89
6	232.9991416	233	233
7	609.9996721	610	610
8	1596.9998748	1597	1597
9	4180.9999522	4181	4181
10	10945.9999817	10946	10946
11	28656.999993	28657	28657
12	75024.9999973	75025	75025
13	196417.999999	196418	196418
14	514228.9999996	514229	514229
15	1346268.9999999	1346269	1346269

$\Phi = 1.618034$

$\Phi + \phi = 1$

D(n) uses initial values 1 and 2

$\phi = -0.618034$ note minus sign

$\sqrt{5} = 2.236068$

FIBONACCI TRIANGLE

$F(n) = F(n-1) + F(n-2)$

FACTORIALS

$f(n) = n\,f(n-1)$

↗ D more F's

↑ more F's

34

13 21

5 8 13 21 34 55

2 3 5 8 13 21 34

↑

F = 1 1 2 3 5 8 13 21 34 55 89 144 233

↓

0 1 1 2 3 5 8 13

Δ -1 0 1 1 2 3 5

1 1 0 1

0 -1 1

-1 0

1

↓ more F's

Fibonacci Sequence $F_n + F_{n+1} = F_{n+2}$ [1, 1]

1, 1, 2, 3 ...

Lucas Sequence $L_n + L_{n+1} = L_{n+2}$ [1, 3]

1, 3, 4, 7 ...

	N	1	2	3	4	5	6	7			
F		2,	3,	5,	8,	13,	21,	34,	55,	89,	144 ...
L		1,	3,	4,	7,	11,	18,	29,	47,	76,	123 ...

δ 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, ... = F   (F − L ≈ F)

PADOVIAN NUMBERS

$F_n + F_{n+1} = F_{n+3}$

$r^3 - r - 1 = 0$

1 2

PRODSUM NUMBERS

PRODSUM1.WPD October 6, 2005

PRODSUM PAIRS PART I

Prodsum pairs are pairs of numbers whose sum is equal to their product. There are two basic questions associated with prodsum numbers:

1) Given any real number x, what is its prodsum partner y, such that $x+y = x\cdot y$?
   It immediately follows that if $x+y = x\cdot y$, then $y = x/(x-1)$ and $x = y/(y-1)$
   Further: let $p = x+y = x\cdot y$. then $p = x^2/(x-1) = y^2/(y-1)$

2) Given any real number p, what are the prodsum numbers x and y such that
   $$x + y = x \cdot y = p\ ?$$
   From $p = x^2/(x-1)$, we have $x^2 - p\cdot x + p = 0$ note[1]
   The two roots of this equation are $x = [p + \sqrt{(p^2-4p)}]/2$ and $y = [p-\sqrt{(p^2-4p)}]/2$

Some examples of prodsum pairs:

$$x = \pi = 3.141592....,\qquad y = \pi / (\pi -1) = 1.466942 ....$$
$$(3.141592) + (1.466942) = 4.608534 \text{ and } (3.141592)\cdot(1.466942) = 4.608534$$

$$x = e = 2.718282....,\qquad y = e / (e -1) = 1.581977....$$
$$(2.718282) + (1.581977) = 4.300259 \text{ and } (2.718282)\cdot(1.581977) = 4.300259$$

$$x = \Phi = 1.618034...,\qquad y = \Phi/ (\Phi -1) = 2.618034....$$
$$(1.618034) + (2.618034) = 4.236068 \text{ and } (1.618034)\cdot(2.618034) = 4.236068$$

$$x = 5 - \sqrt{15} = 1.127017....,\qquad y = 5 + \sqrt{15} = 8.872983....$$
$$(1.127017) + (8.872983) = 10 \quad\text{and}\quad (1.127017) \cdot (8.872983) = 10$$

Properties of prodsum pairs:

$$x + y = x \cdot y = p$$

$$x = y/(y - 1),\qquad y = x/(x - 1)$$

$$p = y^2/(y - 1) = x^2/(x - 1)$$

$$x = [p + \sqrt{(p^2 - 4p)}]/2,\qquad y = [p - \sqrt{(p^2 - 4p)}]/2$$

$$x^2 + y^2 = p^2 - 2p,\qquad x^2 - y^2 = p \sqrt{(p^2 - 4p)}$$

---

[1] This quadratic equation is the characteristic equation for the recursive equation $A_{n+2} = p (A_{n+1} - A_n)$, which has the explicit solution $A_n = (x^n - y^n)/(x - y)$

PRODSUM2.WPD October 9, 2005

PRODSUM PAIRS PART II

$x^2 + bx + c = 0$
$b = -c$ for prodsum

Prodsum pairs are pairs of numbers [x,y] whose sum is equal to their product:

$$x + y = x \cdot y.$$

If $p = x + y = x \cdot y$, then x and y in terms of p are given by the roots of the equation

$$x^2 - px + p = 0,$$

Namely, $x = [p + \sqrt{(p^2 - 4p)}]/2$ and $y = [p - \sqrt{(p^2 - 4p)}]/2$

TABLE 1. gives the values of the prodsum pairs corresponding to some integer values of p.

TABLE 1.

p	y	x	$y^2/2$	$x^2/2$	$\sqrt{}$
-6	-6.872983	0.872983	23.618947	0.381050	60
-5	-5.854102	0.854102	17.135255	0.364745	45
-4	-4.828427	0.828427	11.656853	0.343146	32
-3	-3.791288	0.791288	7.186932	0.313068	21
-2	-2.732051	0.732051	3.732051	0.267949	12
-1	-1.618034	0.618034	1.309017	0.190983.	5
+4	2	2	2	2	0
+5	1.381966	3.618034	0.954915	6.545085	5
+6	1.267949	4.732051	0.803847	11.196153	12
+7	1.208712	5.791288	0.730492	16.769508	21
+8	1.171573	6.828427	0.686291	23.313708	32
+9	1.145898	7.854101	0.656541	30.843459	45
+10	1.127017	8.872983	0.635083	39.364917	60

- The $\sqrt{}$ column gives the values of $\sqrt{(p^2 - 4p)}$
- The values of x and y are imaginary for p = 1, 2, and 3 and = 0 for p = 0.
- Note that for all p, $x + y = x \cdot y = p$; $(x^2 + y^2)/2 = p \cdot (p-2)/2$; and $x^2 \cdot y^2/4 = p^2/4$
- For p = +5, $y = 3 - \Phi$ and $x = 2 + \Phi$ where Φ is the golden ratio.
- When p = - 1, the values are those of the Fibonacci numbers and the golden ratio.

$L = +10$

$$\lim \frac{T_{n+1}}{T_n} = 8.872983$$

$$\lim \frac{T_n}{T_{n+1}} = \frac{1}{100} \times 8.872983$$

PRODSUM4.WPD November 30, 2005

PRODSUM PAIRS PART III

Measured values:

$\alpha = 0.00729735308$ $\log \alpha = -2.13683464$

$\mu = 1836.1526675$ $\log \mu = 3.26390879$

$A = \log_{10}(\alpha\mu) = 1.12707415$

Prodsum pairs:

$J = 5 - \sqrt{15} = 1.12701665379258311482073460021 76$

$K = 5 + \sqrt{15} = 8.87298334620741688517926539978 24$

$J+K = J \cdot K = 10$

$1/K = 0.112701665379258311482073460021 76$

$10 \cdot 1/K = J = 1.12701665379258311482073460021 76$

The convergence of the measured value A to the prodsum value J per iteration of the formula:

$$X_2 = (X_1)^2/10 + 1$$

A =	1.12707415
B = (A²/10) + 1 =	1.12702961395982225
C = (B²/10) + 1 =	1.1270195750742425967853837251595
D = (D²/10 + 1 =	1.1270173122600526344757431027 46
"	1.127016802213387298623836841441 6
"	1.127016687247128934581044533523 7
"	1.127016661333349283528780979665
"	1.127016655492296931404981981903 4
"	1.127016654175704270703997075583 1
"	1.127016653878939899455553000784 2
"	1.127016653812048221711764307727 1
"	1.127016653796970614747581139602 4
"	1.127016653793572071917950483237 2

the reverse =?

* *

J = 1.12701665379258311482073460021 76

Most mathematical sequences, series, continued fractions, etc. are generated by "dialectical processes" consisting of <u>two altering operations</u>, such as the above divide-add combination.

Old G?

PRODSUM3.WPD October 17, 2005

THIS SCRAP POINTS OUT PARALLELS BETWEEN A RECURSION EQUATION AND THE VALUES OF THREE FUNDAMENTAL CONSTANTS OF PHYSICS.

$A_{n+2} = 10A_{n+1} - 10A_n$ has the charasteric equation, $x^2 - 10x + 10 = 0$, whose solutions are $x = 5 - \sqrt{15}$ and $y = 5 + \sqrt{15}$; with numerical values : $x = 1.1270167$ and $y = 8.8729833$

$y^2/2 = 39.3649167$

$x + y = 10$

$x \cdot y = 10$

$y - x = \sqrt{60} = 7.7459667$

$(x + y) / (x \cdot y) = 1$

$(x^2 + y^2)/2 = 40$

$x^2/2 \cdot y^2/2 = 25$

$\log_{10}$ values of three fundamental constants:
The fine structure constant $\alpha = -2.1368346$
The proton/electron mass $\mu = +3.2639088$
with $\alpha\mu = 1.1270742$ 471 old G?
The coulomb/gravity force $S = 39.3558802$
with $\sqrt{(2S)} = 8.8719649$

$\alpha\mu + \sqrt{(2S)} = 9.9990391$

$\alpha\mu \cdot \sqrt{(2S)} = 9.9993627$

$\sqrt{(2S)} - \alpha\mu = 7.7448907$

$[\alpha\mu + \sqrt{(2S)}] / [\alpha\mu \cdot \sqrt{(2S)}] = 1.0000324$

$(\alpha\mu)^2/2 + S = 39.9910283$

$(\alpha\mu)^2/2 \cdot S = 24.9968136$

$\alpha\mu/x = 1.0000510$

$(y^2/2) / S = 1.0002296$

$y / \sqrt{(2S)} = 1.0001148$

$\alpha\mu - x = 0.0000575$

$y^2/2 - S = 0.0090365$

$y - \sqrt{(2S)} = 0.0010184$

The explicit formula for the values of A_n is

$A_n = (y^n - x^n)/(y - x)$

This formula leads to the following series:

$A_0 = 0$

$A_1 = 1$

$A_2 = 10$

$A_3 = 90$

$A_4 = 800$

An explicit formula for the values of C_n is

$C_n = \{[\sqrt{(2S)}]^n - (\alpha\mu)^n\}/[\sqrt{(2S)} - \alpha\mu]$

giving the following series:

$C_0 = 0$

$C_1 = 1$

$C_2 = 9.9990390$

$C_3 = 89.9814202$

$C_3 = 799.7437196$

The many parallels between the fundamental physical constants $\alpha\mu$ and S with the solutions of the recursive equation $A_{n+2} = 10A_{n+1} - 10A_n$ suggest that some form of "continental drift" may have occurred. It has been proposed by several [see Dirac. 1935] that the fundamental constants do vary in time. It may be that the original values of $\alpha\mu$ and S were 1.1270167 and 39.3949167, respectively and have drifted over 13 billion years to their present values of 1.1270742 and 39.3558802. [$\alpha\mu$ increasing and S decreasing]. Although the drift is slow, is it possible to predict their limit points ?

OR $\alpha\mu$, is measured in earth laboratories. What would be the value in space?

RECURMEAS.WPD January 24, 2006

RECURSIVE ALTERNATIVES UPDATE G

Measured values: (All $\log_{10}$ cgs) $S = -39.355880$ (471) $= G\,m_p^2 / \hbar c\alpha\mu$
$\alpha\mu = 1.1270742$ (115) $m_p^2 = -47.553204$ $\hbar c = -16.500103$ $G = -7.175705$ update 296

The general characteristic polynomial corresponding to the recursion equation

$$\mathbf{A_{n+2} = bA_{n+1} + cA_n} \quad \text{is}$$

$x^2 - bx - c = 0$, with roots $v = (b+\sqrt{b^2+4c})/2$ and $u = (b-\sqrt{b^2+4c})/2$

$u+v = b$ and $u \cdot v = -c$

CASE 1) the "ΠΣ" Case Assume: the characteristic polynomial is $x^2 - 10x + 10 = 0$,
$v = 5 + \sqrt{15} = 8.8729805$, $u = 5 - \sqrt{15} = 1.1270167$, and $v^2/2 = 5(4+\sqrt{15}) = 39.364892$
$u + v = 10.0000000$, $u \cdot v = 10{,}0000000$

CASE 2) Assume: $u = \alpha\mu = 1.1270742$ and $v^2/2 = S = 39.355880$, then $v = 8.8719648$
$u + v = 9.9990390$, $u \cdot v = 9.9993626$

CASE 3) Assume: $u = \alpha\mu = 1.1270742$ and that $u + v = 10$
then $v = 8.8729258$, and $v^2/2 = 39.364406$, and $u \cdot v = 10.0004457$
$u + v = 10.0000000$, $u \cdot v = 10.0004457$

CASE 4) Assume: $u = \alpha\mu = 1.1270742$ and that $u \cdot v = 10$
then $v = 8.8725303$, and $v^2/2 = 39.360897$, and $u + v = 9.9996045$
$u + v = 9.9996045$, $u \cdot v = 10.0000000$

Assume in CASES 2), 3), and 4) that the differences between the $v^2/2$ values and the ΠΣ value of 39.364892 are attributable to an error in G. Then the "correct" values of G to make the difference = 0 and their differences, Δ, with the present value of $G = -7.175705$ are:

CASE 2) $G = -7.184717$, $\Delta = 0.009012$
CASE 3) $G = -7.184231$, $\Delta = 0.008526$
CASE 4) $G = -7.180722$, $\Delta = 0.005017$
5)
6)
7)

RECURTBL.WPD January 24, 2006

RECURSION TABLE

$A_{n+2} = b\,A_{n+1} - c\,A_n \qquad x^2 - b\,x + c = 0$

	CODE	FORM	$b = u + v$	$c = u \cdot v$	u	v	$v^2/2$	$\Delta(10 - b)$	$\Delta(10 - c)$
	ΠΣ	b = c	10	10	$5 - \sqrt{15}$	$5 + \sqrt{15}$	$5(4 + \sqrt{15})$		
1	ΠΣ	b = c	10	10	1.1270167	8.8729805	39.364892	0	0
2	αμ, S	measured	9.9990390	9.9993627	1.1270742	8.8719648	39.355880	0.0009610	0.0006373
3	αμ, b		10	10.0004457	1.1270742	8.8729258	39.364406	0	0.0004457
4	αμ, c		9.9996045	10	1.1270742	8.8725303	39.360897	0.0004955	0
	S, b		10	10.007889	1.1280352	8.8719648	39.355880	0	0.007889
	S, c		9.9991108	10	1.127146	8.8719648	39.355880	0.0008892	0
5	αμ	b = c	9.9964922	9.9964922	1.1270742	8.869418	39.333288	0.0035078	0.0035078
	S	b = c	9.9989979	9.9989979	1.1270331	8.8719648	39.355880	0.0010021	0.0010021
6	$(\alpha\mu)^2/10 + 1$	b = c	9.9992106	9.9992106	1.1270296	8.8721810	39.357798	0.0007894	0.0007894
7	$[10(\alpha\mu - 1)]^{1/2}$	b = c	9.9844646	9.9844646	1.1272719	8.8571927	39.224931	0.0155354	0.0155354
	$\Phi^{1/4}$	b = c	9.9502081	9.9502081	1.1278385	8.8223696	38.917103	0.0497919	0.0497919

Inputs are in red. When b = c, $u = v/(v-1)$ and $v = u/(u-1)$, And $b = c = v^2/(v-1) = u^2/(u-1)$