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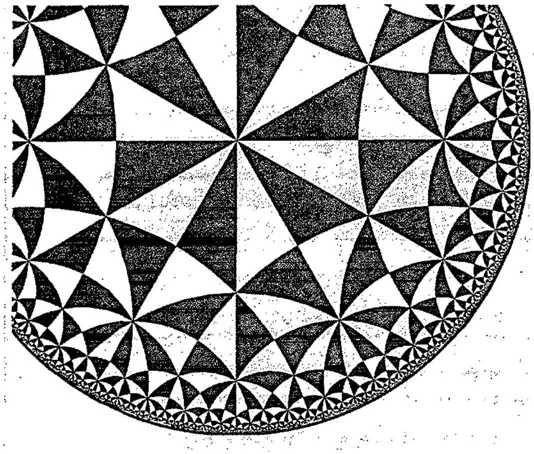
Visions of Infinity

Tiling a hyperbolic floor inspire both mathematics and art

By IVARS PETERSON



Dunham

A circular pattern of triangles with curved sides, similar to this one, inspired M.C. Escher to create his Circle Limit *series of prints.*

Even the most brilliant innovators get their inspiration from somewhere.

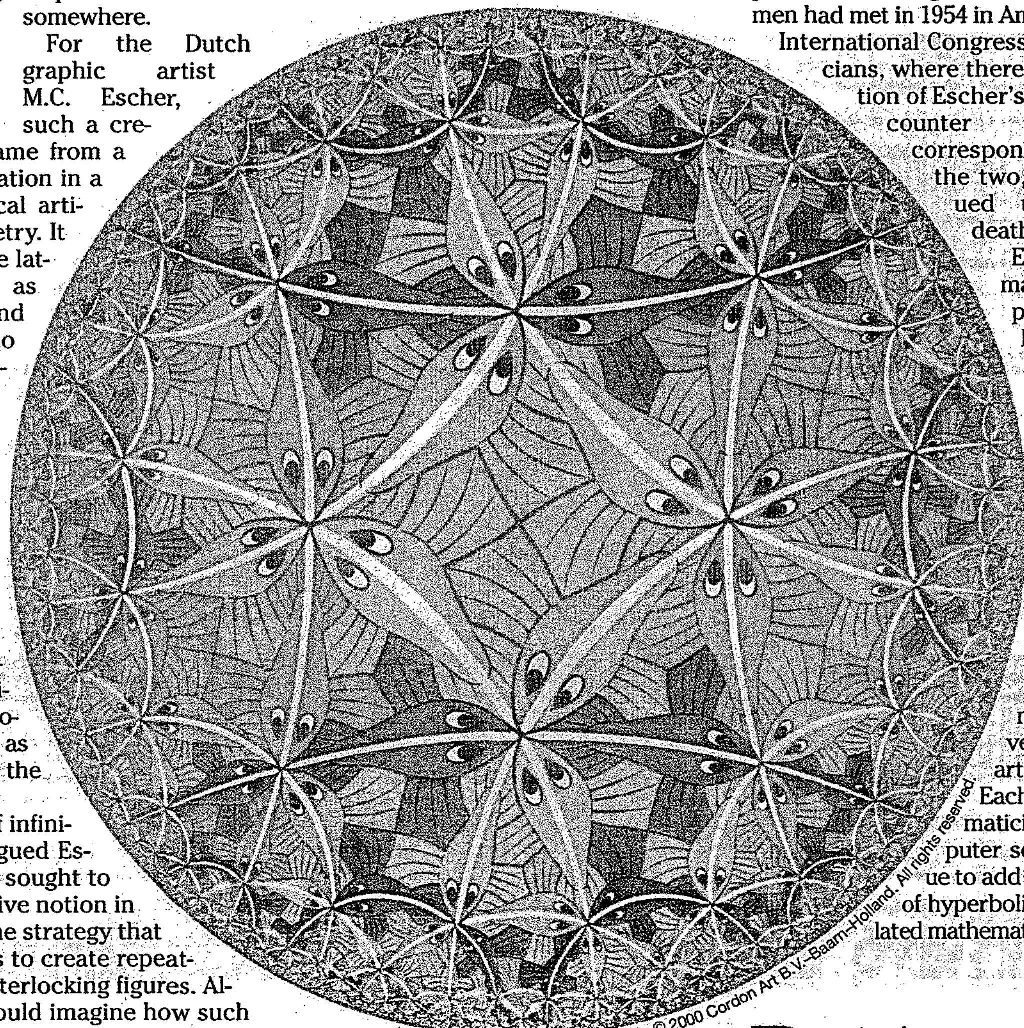
For the Dutch graphic artist M.C. Escher, such a creative impetus came from a particular illustration in a 1957 mathematical article about symmetry. It gave him what he later described as "quite a shock" and inspired him to create four artworks: the *Circle Limit* series of prints.

The illustration showed a curious tiling of black and white triangles with curved sides. Enclosed within a circle, the alternately colored triangles became progressively smaller as they approached the circle's perimeter.

The concept of infinity had long intrigued Escher, and he had sought to capture this elusive notion in visual images. One strategy that he employed was to create repeating patterns of interlocking figures. Although Escher could imagine how such arrays could extend infinitely, the actual patterns he drew, of course, represented only a fragment of an infinite expanse.

In another approach, Escher tried to fit together replicas of a figure, such as a fish, that diminish in size as they spiral into or recede from a point in the middle of a square or circular frame. However, he wasn't entirely satisfied with these efforts.

The mathematical drawing—an illustration of the so-called hyperbolic plane—that had so startled Escher offered him a precise, aesthetically pleasing way to depict diminishing figures within a circle.



M.C. Escher's "Circle Limit III."

The article containing the diagram had been written by H.S.M. Coxeter, a mathematician at the University of Toronto. Today in his nineties, Coxeter continues to focus on the interplay of symmetry and geometric shapes. His research covers topics such as the mathematics of kaleidoscopic patterns.

Coxeter had sent Escher a copy (symmetry article as a thank you fo mission to reproduce several of Esc periodic drawings as illustrations. men had met in 1954 in Amsterdam a International Congress of Mathe cians, where there was an e: tion of Escher's work. Th counter led correspondence bet the two, which co ued until Esc death in 1972.

Escher in mailed Coxe print of " Limit I" first fruit artist's ture, ins by Co into h bolic ge try. In decades Escher's tial fora *Circle* prints hav tivated mathemati vestigations artistic endea

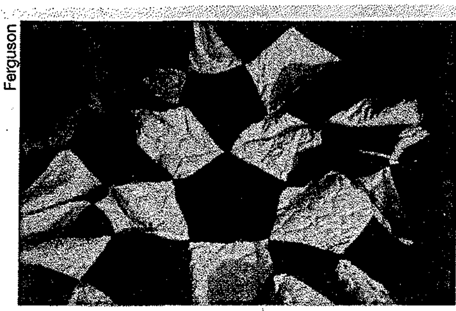
Each year, m maticians and puter scientists c ue to add to the liter of hyperbolic tilings a lated mathematical topics.

Despite the many manifestatic Escher-inspired research, stems from the same basic p ples of curved geometry.

For example, if you draw any tri on a sheet of paper and add up its angles, the result is always 180 deg When you draw a triangle on a sa shaped surface, however, the angl variably add up to less than 180 deg

Just as a flat surface—like that of a

Ferguson



A hyperbolic quilt sewn together from pentagons of cloth, with four pentagons meeting at each corner in the pattern.

of paper—is a piece of the infinite mathematical surface known as the Euclidean plane, a saddle-shaped surface can be thought of as a small piece of the hyperbolic plane. Picturing what the hyperbolic plane looks like on a larger scale, however, requires some mind-bending ingenuity.

Freelance mathematician Jeffrey R. Weeks of Canton, N.Y., suggests making "hyperbolic paper" from a large number of equilateral triangles as one way to get a feel for the hyperbolic plane.

Taping together equilateral triangles so that precisely six triangles meet at each vertex produces a flat sheet. In contrast, assembling equilateral triangles so seven triangles meet at each vertex produces a floppy, bumpy surface. The more triangles you use and the larger the resulting sheet, the more closely it resembles the hyperbolic plane.

A similar construction can be done with pentagons. Mathematician and sculptor Helaman Ferguson of Laurel, Md., has made an intriguingly wrinkly hyperbolic quilt by sewing together pentagonal patches of fabric so that four pentagons meet at each corner. It's an unruly blanket that refuses to lie flat, he says.

Such constructions are not the only way to visualize the hyperbolic plane. More than a century ago, French mathematician Henri Poincaré introduced a method for representing the entire hyperbolic plane on a flat, disk-shaped surface.

In Poincaré's model, the hyperbolic plane is compressed to fit within a circle. The circle's circumference represents points at infinity. In this context, a straight line, meaning the shortest distance between two points, is a segment of a circular arc that meets the Poincaré disk's circular boundary at right angles

Although this model distorts distances, it represents angles faithfully. The hyperbolic measure of an angle is equal to that measured in the disk representation of the hyperbolic plane. So, a repeating pattern made up of identical geometric shapes in the hyperbolic plane transforms, when represented in a Poincaré model, into an array of shapes that diminish in size as they get closer to the disk's bounding circle.

For his *Circle Limit* prints, Escher worked out the underlying rules of these disk models and developed his own method for constructing a hyperbolic grid, relying on his skill and intuition to create the geometric scaffolding he needed. Coxeter's "hocus pocus" mathematical text wasn't much help, Escher later remarked in a letter to Coxeter. Nonetheless, Escher executed the drawings with extraordinary accuracy, Coxeter comments.

"The first time I saw a print of 'Circle Limit III,' I said to myself, 'that is the most beautiful example I have ever seen of the Poincaré circle model for hyperbolic geometry,'" says J. Taylor Hollist, a mathematician at the State University of New York at Oneonta. He's documented many historical interactions between Escher and scientists.

Besides its beauty, "Circle Limit III" also presents a puzzle. For some reason, in this particular case, Escher drew a pattern of lines somewhat different from that in Coxeter's original drawing. The main arcs seen in "Circle Limit III" meet the circumference at a specific angle very close to 80 degrees rather than precisely 90 degrees.

Coxeter was able to demonstrate that each arc is of a type known to mathematicians as an equidistant curve. It bears the same relationship to a hyperbolic straight line as a line of latitude does to the equator on the surface of a sphere.

When Coxeter worked out trigonometrically what the proper angle of such a curve in Escher's print should be, he obtained 79 degrees 58 minutes, again confirming the accuracy of Escher's draftsmanship.

Besides intriguing professional mathematicians, Escher's *Circle Limit* prints and their repeating patterns prove to be useful vehicles for becoming comfortable with and teaching hyperbolic geometry, says Douglas J. Dunham of the computer science department at the University of Minnesota at Duluth. "Even to mathematicians, hyperbolic geometry is not that familiar," he contends.

Dunham and his students have written several computer programs to generate hyperbolic patterns, particularly those made up of repeating motifs colored in various ways.

Mathematicians use a standard notation for describing a mosaic made up of identical tiles, where each tile is a polygon with a given number of edges of the same length and the same number of corners or vertexes. On a flat surface, there are three such tilings. A tiling in which six equilateral triangles meet at each intersection is designated {3,6}, one in which four squares meet at each intersection is {4,4}, and one in which three hexagons meet at each intersection is {6,3}.

The same notation applies to regular tilings of the hyperbolic plane. A tiling where four pentagons meet at each vertex is labeled {5,4}. In general, for polygons with p sides, meeting q at a vertex, the result is a hyperbolic tiling when $(p-2)$ multiplied by $(q-2)$ is greater than 4.

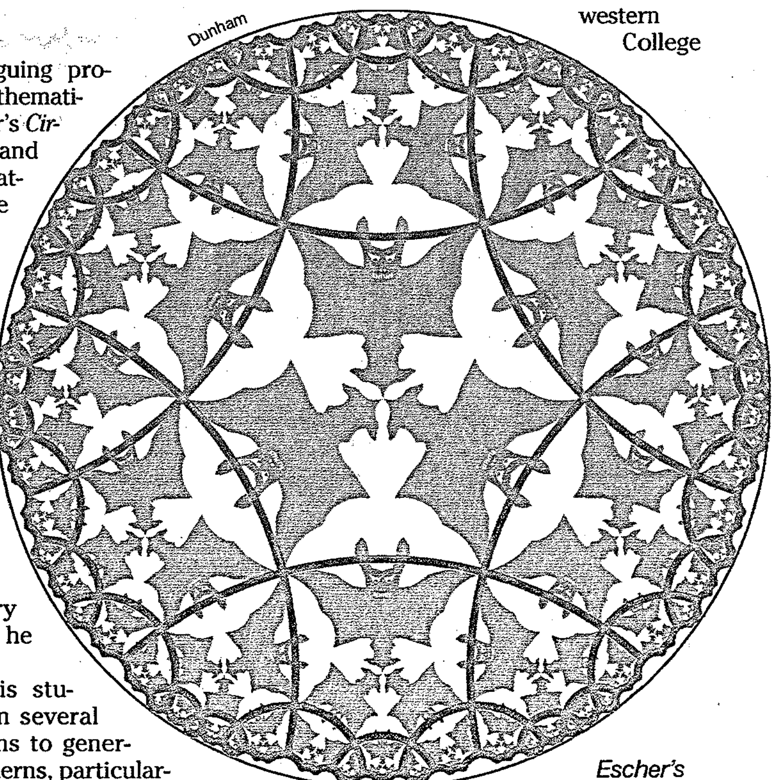
Escher's "Circle Limit IV," which features interlocking devils and angels, is an example of a {6,4} tiling. In other words, the underlying hyperbolic grid consists of hexagons that meet four at each vertex.

Dunham has developed a computer program that transforms a hyperbolic Escher design from one tiling pattern to another. For example, he can transform the {8,3} pattern of crosses in "Circle Limit II" into a strikingly different {10,3} tiling, where the central figure is a star.

The same pattern can be transformed into an array of starbursts with any number of rays, Dunham notes. Indeed, there is an infinite number of hyperbolic tilings available for such transformations. The use of different motifs and color schemes increases the possibilities even further.

At a conference on mathematical connections in art, music, and science held last summer at Southwestern College

Dunham

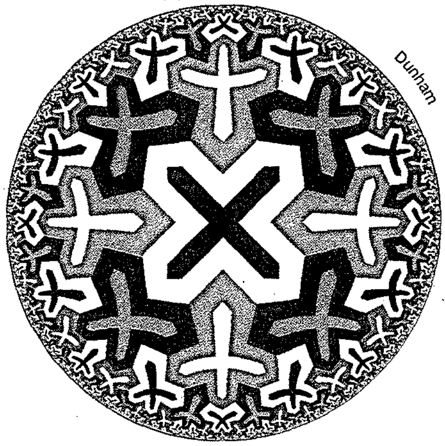


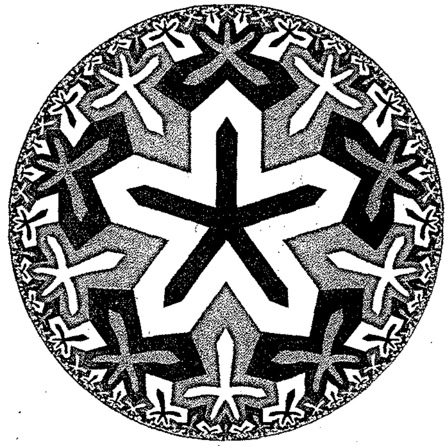
Escher's "Circle Limit IV," as seen in this computer-generated rendition in gray and white, has an underlying tiling pattern in which four hexagons meet at each vertex.

He had worked with Celtic knot patterns, used centuries ago in Ireland and elsewhere to decorate religious texts.

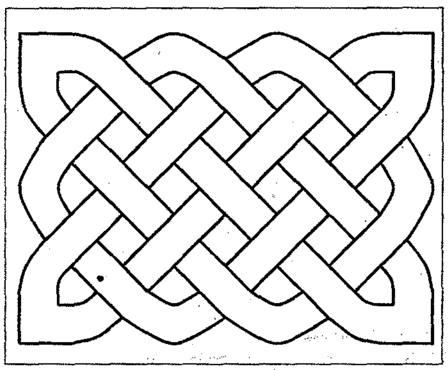
... alternating over-and-under pattern, then joining the two ends to form a continuous band.

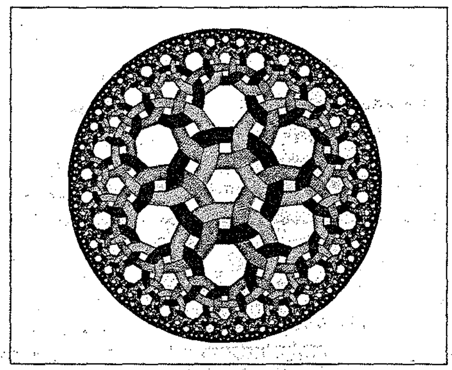
Dunham now has a computer program





Douglas Dunham has developed a computer program that can transform one hyperbolic tiling pattern into another. In this case, he has used the program to convert the crosses of his rendition of Escher's "Circle Limit II" pattern (left) into stars (right).



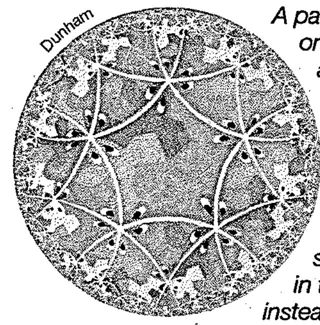


A simple Celtic knot pattern (left). Such a woven pattern can be converted into an array of interlocking rings, then depicted using hyperbolic geometry (right).

sions of these ancient designs. He can also construct examples in which rings interlock in the over-and-under patte characteristic of Celtic knots.

Interestingly, Escher himself incorpor ed an intricate pattern of interlocking rin within a circular frame in his last woodc "Snakes." Dunham can show that this pa tern is closely related to a hyperbolic va ant of a Celtic weaving. Although Esch and Dunham approached these pattern from different perspectives, their intelle tual common ground is apparent.

In a 1960 essay later translated and pu lished in the book *The Graphic Work* *M.C. Escher* (1961, Duell, Sloan and Pearce



A pattern based on transforming a computer-generated rendition of Escher's "Circle Limi III" so that a pentagonal shape appear in the middle instead of a square.

Escher noted, "The ideas that are basic to [my art] often bear witness to my amaze ment and wonder at the laws of nature which operate in the world around us.

"By keenly confronting the enigmas that surround us, and by considering and analyzing the observations that I have made, I ended up in the domain of mathematics," he continued. "Although I am absolutely without training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists." □

Constructing Escher

Nowadays, mathematicians, computer scientists, and others have a variety of speedy computer-based methods for generating hyperbolic patterns and tilings. M.C. Escher didn't have such technology at his disposal. Neither did Henri Poincaré and other 19th-century mathematicians who drew various pictures of the hyperbolic plane. They relied on the traditional tools of geometry—compass and straight-edge—to create their diagrams.

In an article to appear in the January 2001 American Mathematical Monthly, however, Chaim Goodman-Strauss of the University of Arkansas in Fayetteville suggests what these procedural details might have been. He offers techniques and instructions for drawing by hand some tilings of the Poincaré model of the hyperbolic plane.

"Remarkably, this may be the first detailed, explicit synthetic construction of triangle tilings of the hyperbolic plane to appear," Goodman-Strauss notes.

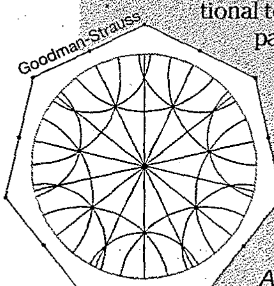
His directions are built out of basic tasks familiar to a student of Euclidean geometry: bisecting a line segment, drawing a parallel through a given point, drawing a perpendicular through a given point, constructing a circle through three given points, and a handful of other operations.

A suitable combination of those activities enables one to construct, for example, the hyperbolic line that passes through two given points. To achieve this, one must create a geometric scaffolding of lines and points outside a Poincaré disk to guide the drawing of arcs and points within the circle's boundary.

Goodman-Strauss worked out the method by extending his expertise in Euclidean geometry to encompass the types of curves and angles necessary to represent hyperbolic structures. He admits that there's probably nothing original in his contribution. "Surely, this was all well-known at the end of the 19th century, just as it has long been forgotten at the dawn of the 21st," he remarks.

Nonetheless, reviving long-lost construction techniques has value. Such exercises offer an illuminating window on not only Escher's art but also the remarkable work of earlier mathematicians who explored non-Euclidean geometries.

"It is wonderfully satisfying to make these pictures by hand, patiently, with pencil and paper, compass and straightedge," Goodman-Strauss adds. "I encourage you to test this theorem for yourself!" —I. P.



A geometric scaffolding of lines and points outside a circle aids the construction of the hyperbolic pattern within the circle.

COMBINATIONS OF POLYGONS WITH THE TILING PROPERTY $\Sigma = 360^{\circ}$

TABLE I $\Sigma = 360^{\circ}$

SIDE	DEGREES					+	+	+		+	+	+	+	+	+	+	+	~~+~~	
3	60	1		1	1		2	3		2			1	1	4		6		
4	90		1			1	1	2			1		2			4			
5	108		1						2										
6	120					1				2		3	1		1			[illegible]	
8	135	1									2								
9	140			1															
10	144				1				1										
12	150					1	1							2					
15	156				1														
18	160			1															
20	162		1																
24	165	1																	

The condition that at a vertex the sum of the angles of the participating polygons be 360° is necessary but not sufficient for the combination to constitute a tiling configuration. Those combinations marked with a + form tiling patterns.

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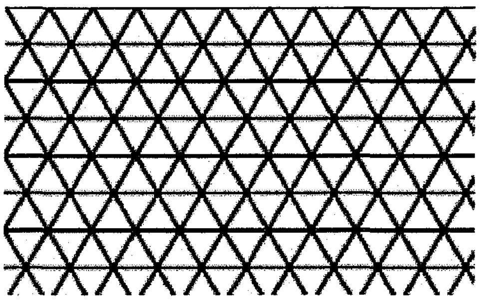
Regular Tessellations (4/4)

The Regular Tessellations

tessellations of a single regular polygon



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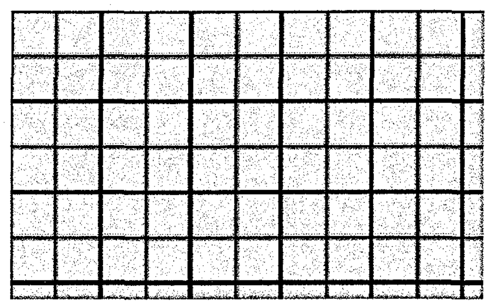

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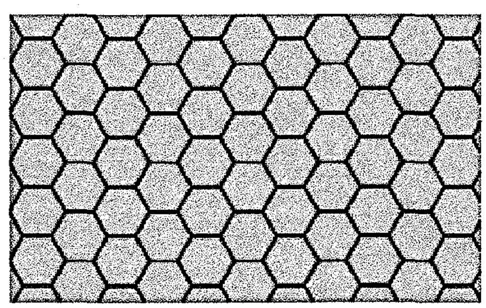


6.6.6

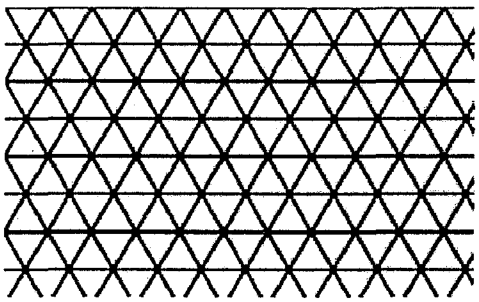
Back Up Next

Real examples of regular tessellations:

Regular Tessellations (4/4)

The Regular Tessellations

tessellations of a single regular polygon



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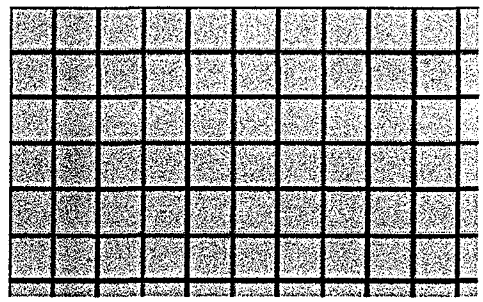

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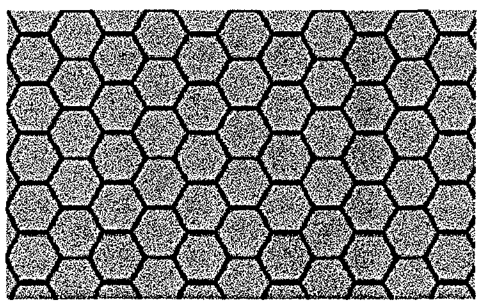

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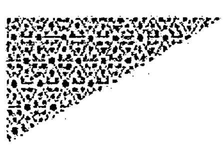
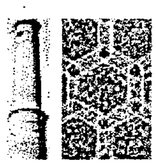
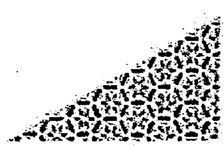
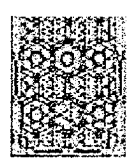
Back Up Next

Real examples of regular tessellations:





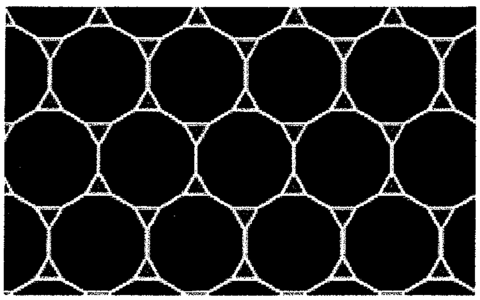




Semiregular Tessellations (4/4)

The Semiregular Tessellations

tessellations of two or more different regular polygons such that the same polygon arrangement exists at every vertex



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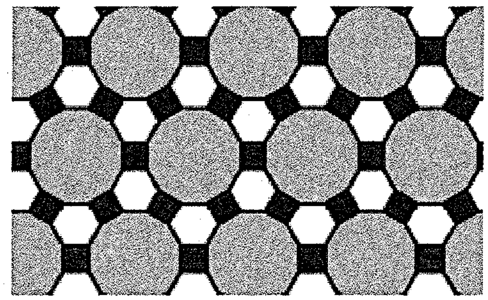

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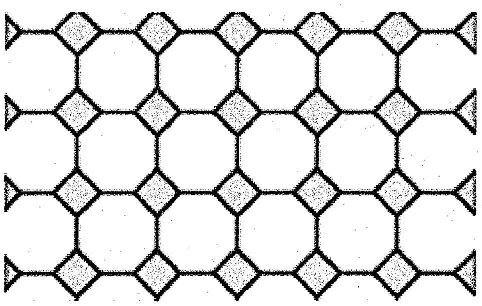

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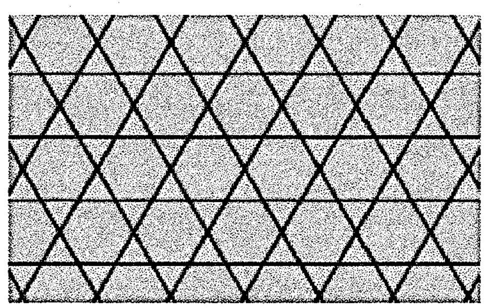

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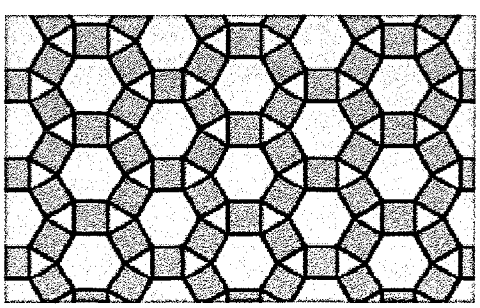

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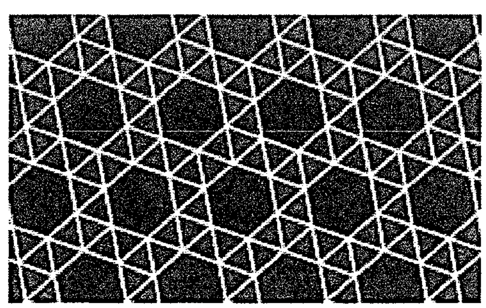

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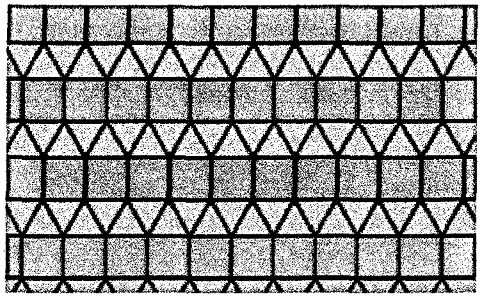

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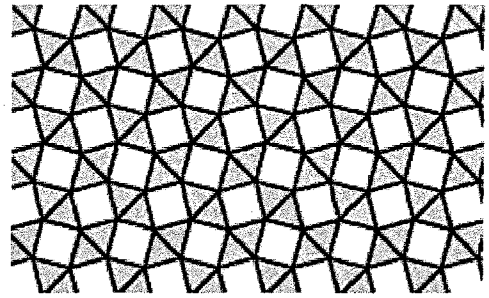

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Back Up Next

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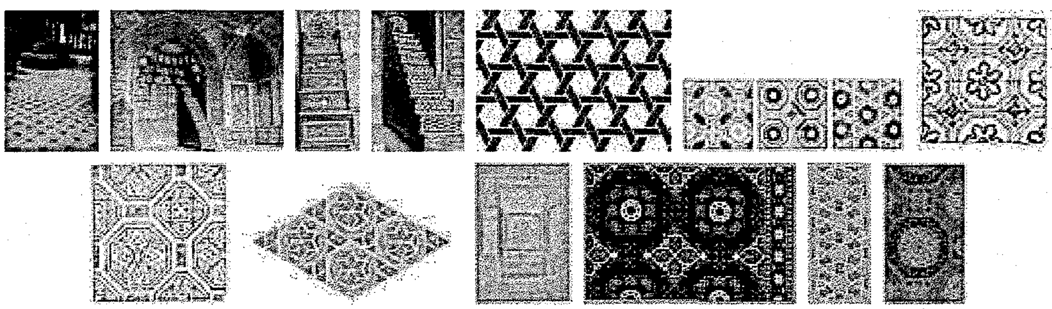

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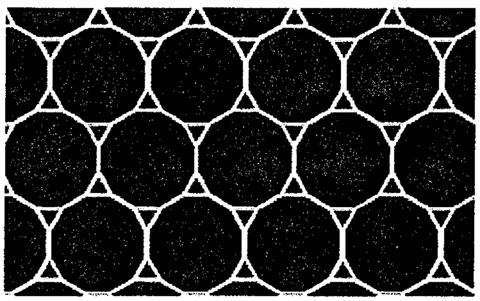
Semiregular Tessellations (4/4)

The Semiregular Tessellations 8

tessellations of two or more different regular polygons such that the same polygon arrangement exists at every vertex



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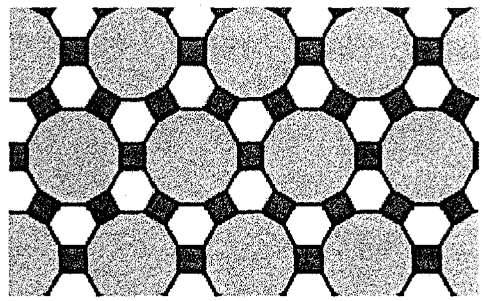

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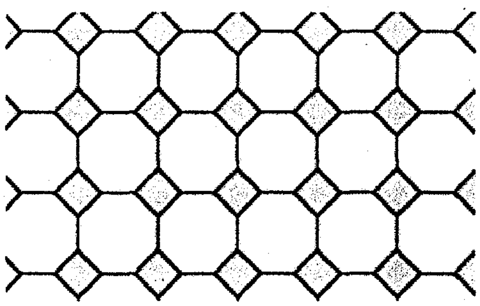

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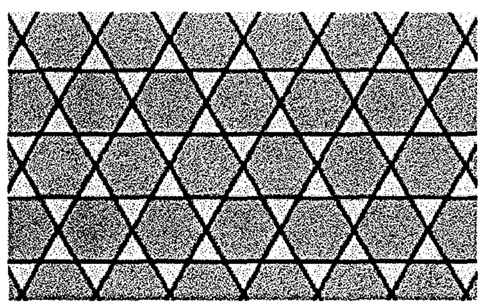

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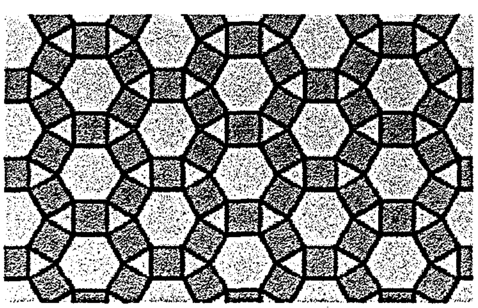

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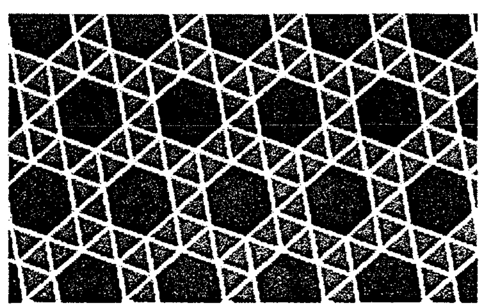

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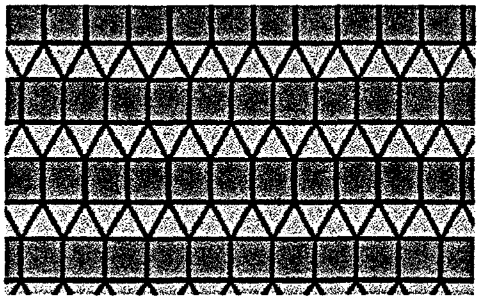

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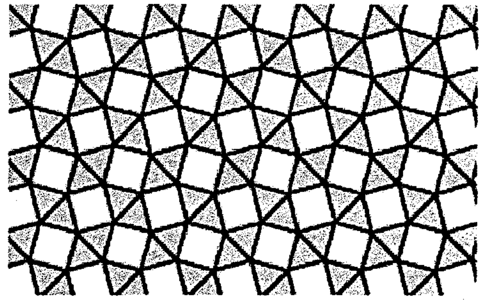

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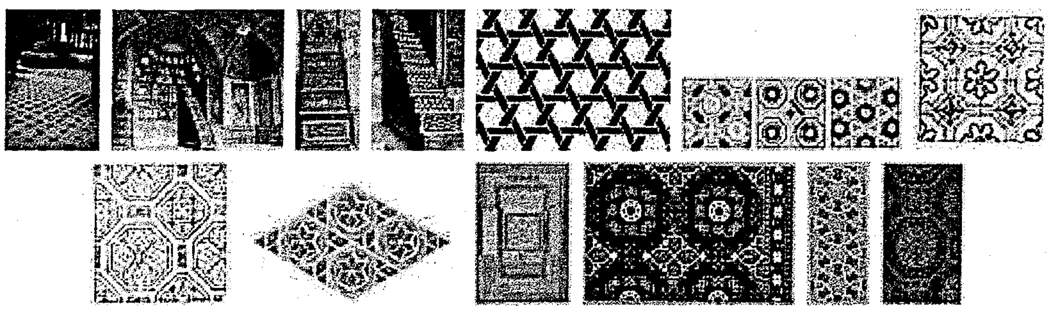

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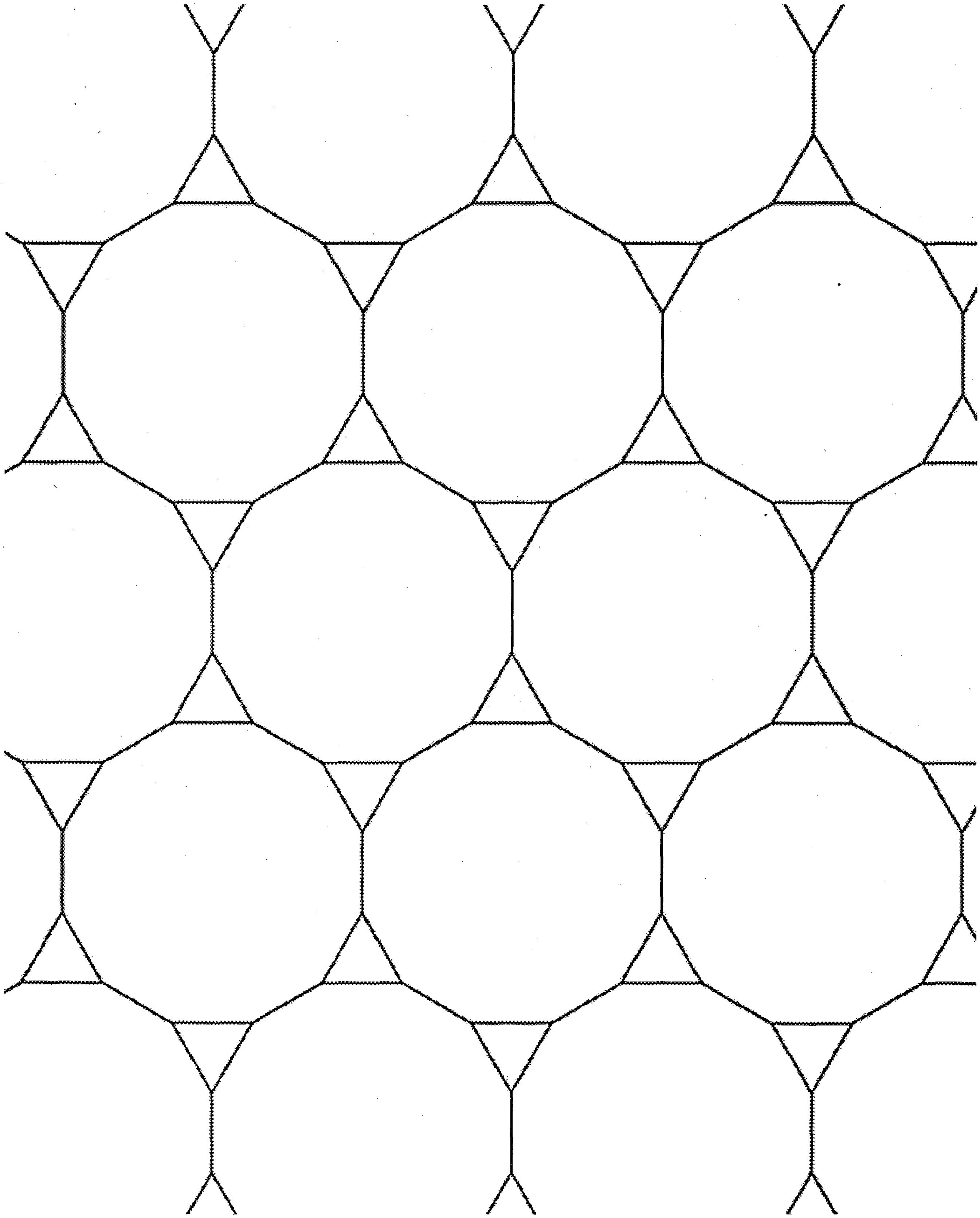

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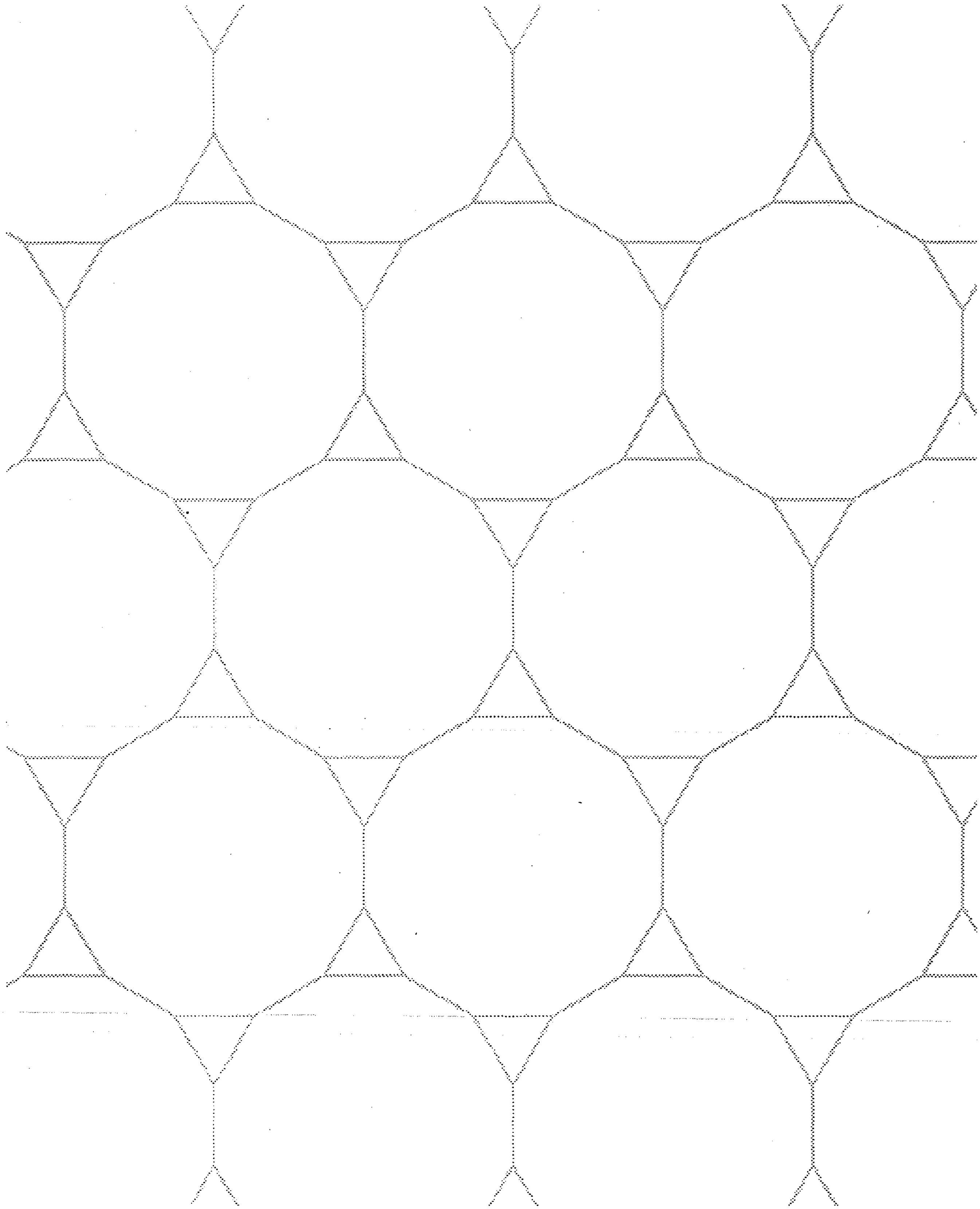

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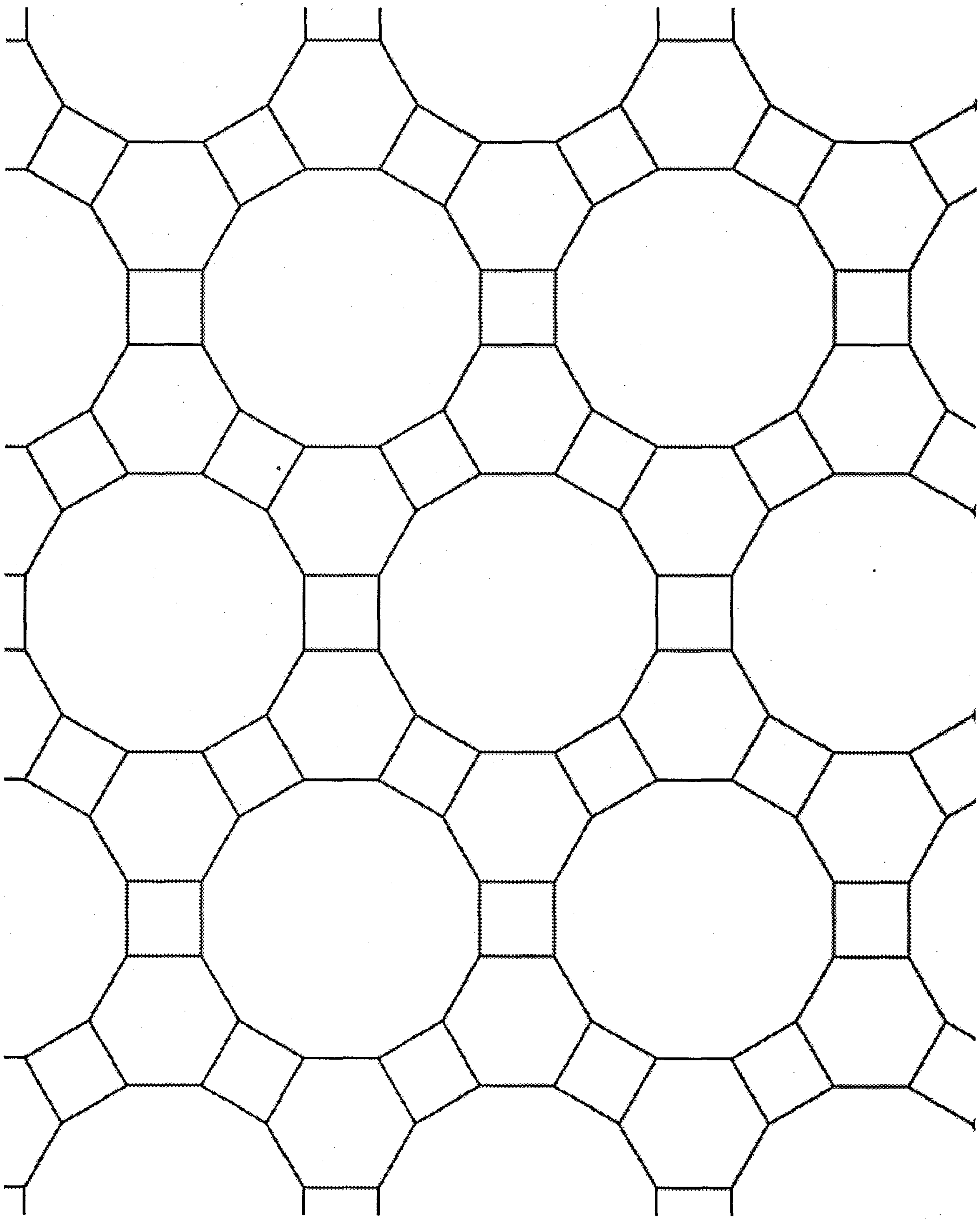

Semiregular Tessellation 4.8.8 (black large image)
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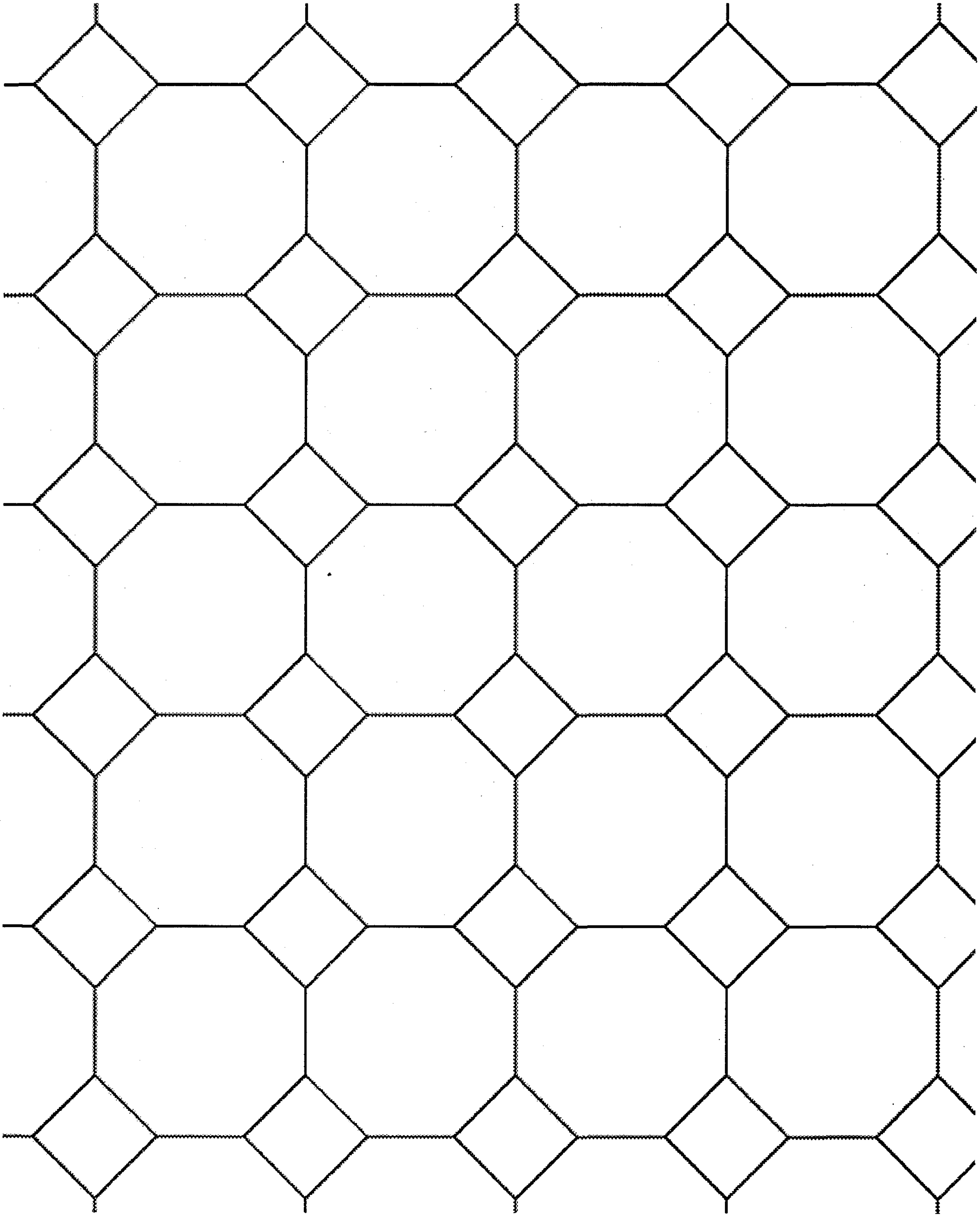

Semiregular Tessellation 3.6.3.6 (black large image)

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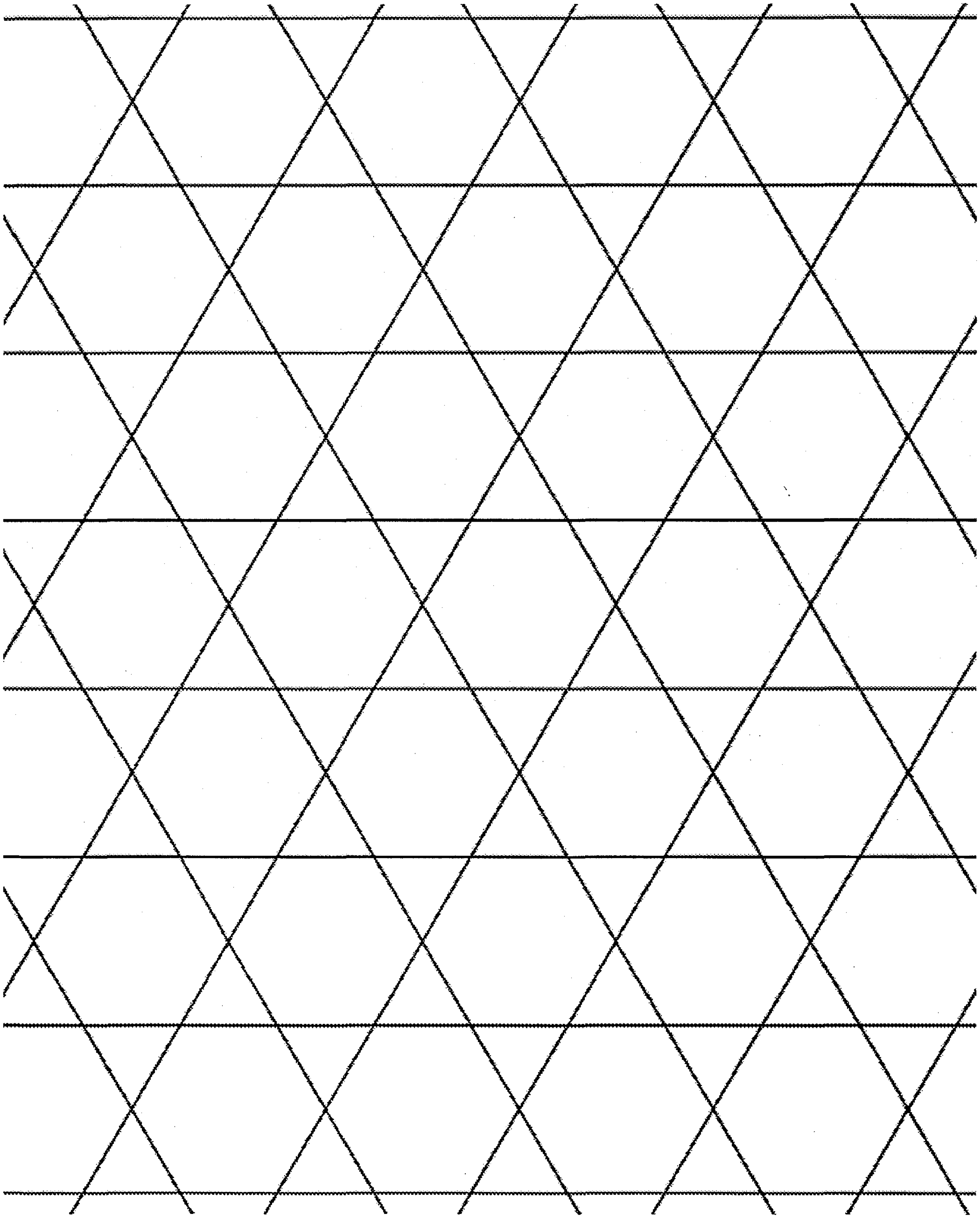

Semiregular Tessellation 3.4.6.4 (black large image)

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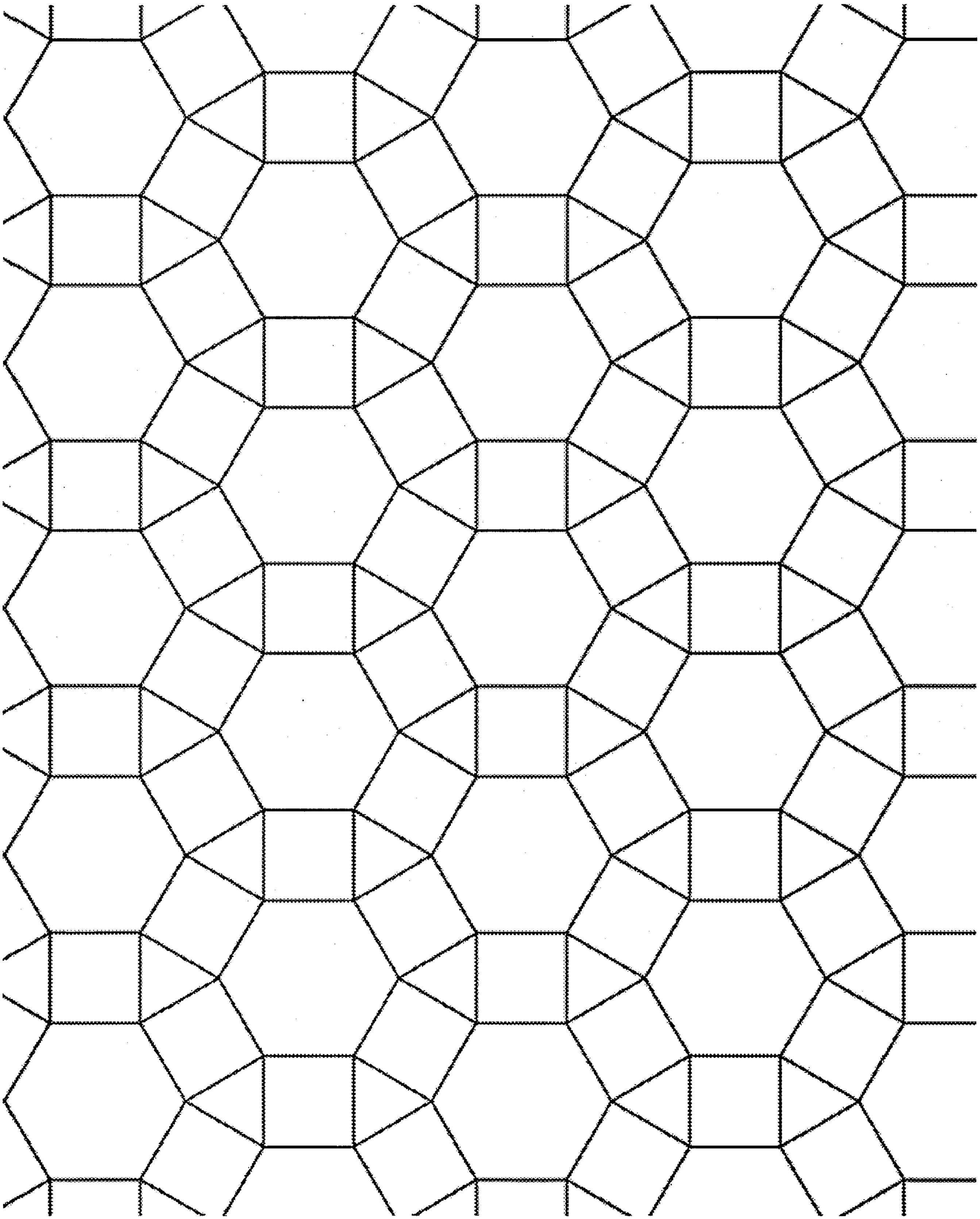

Semiregular Tessellation 3.3.3.3.6 (black large image)

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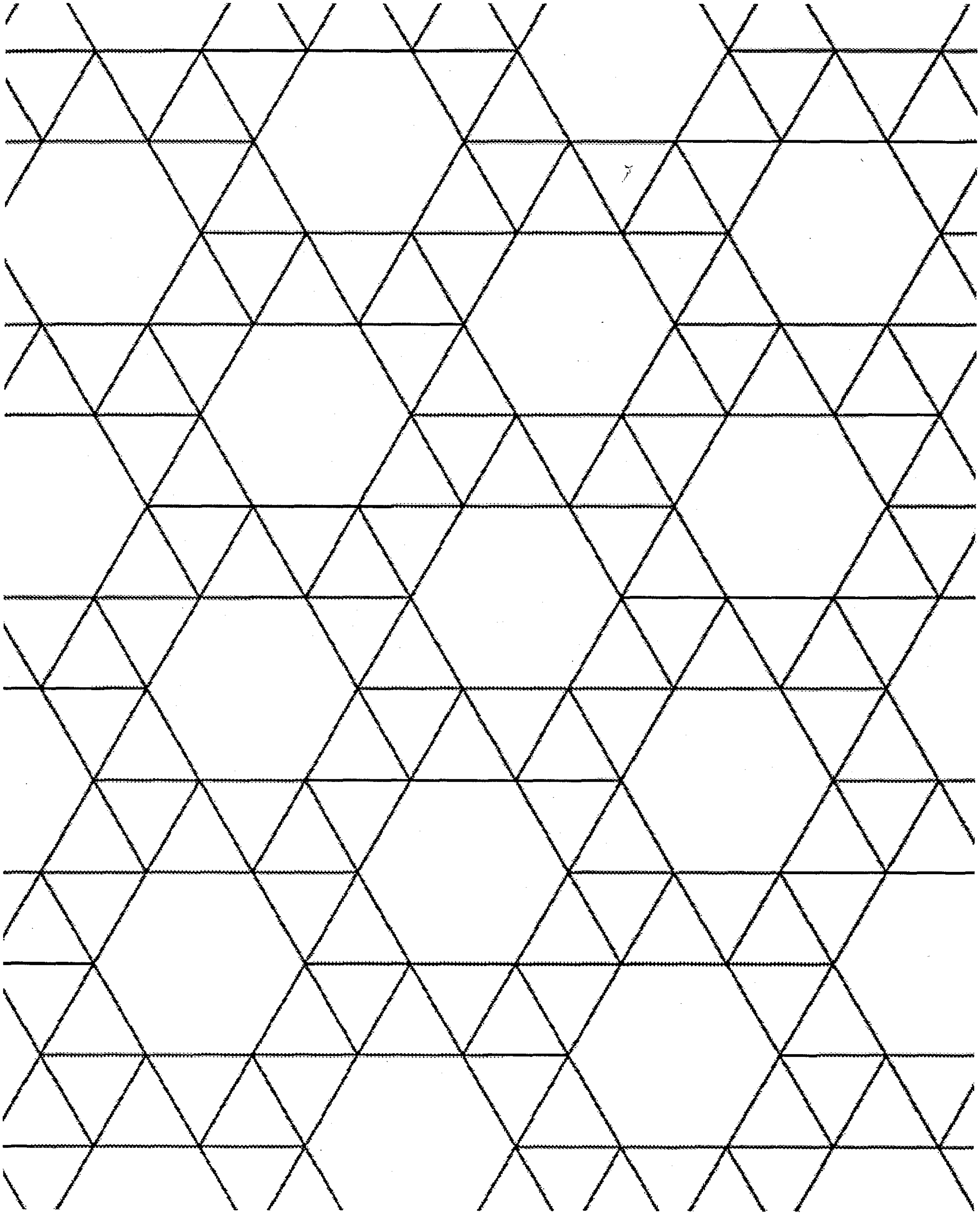

Semiregular Tessellation 3.3.3.4.4 (black large image)

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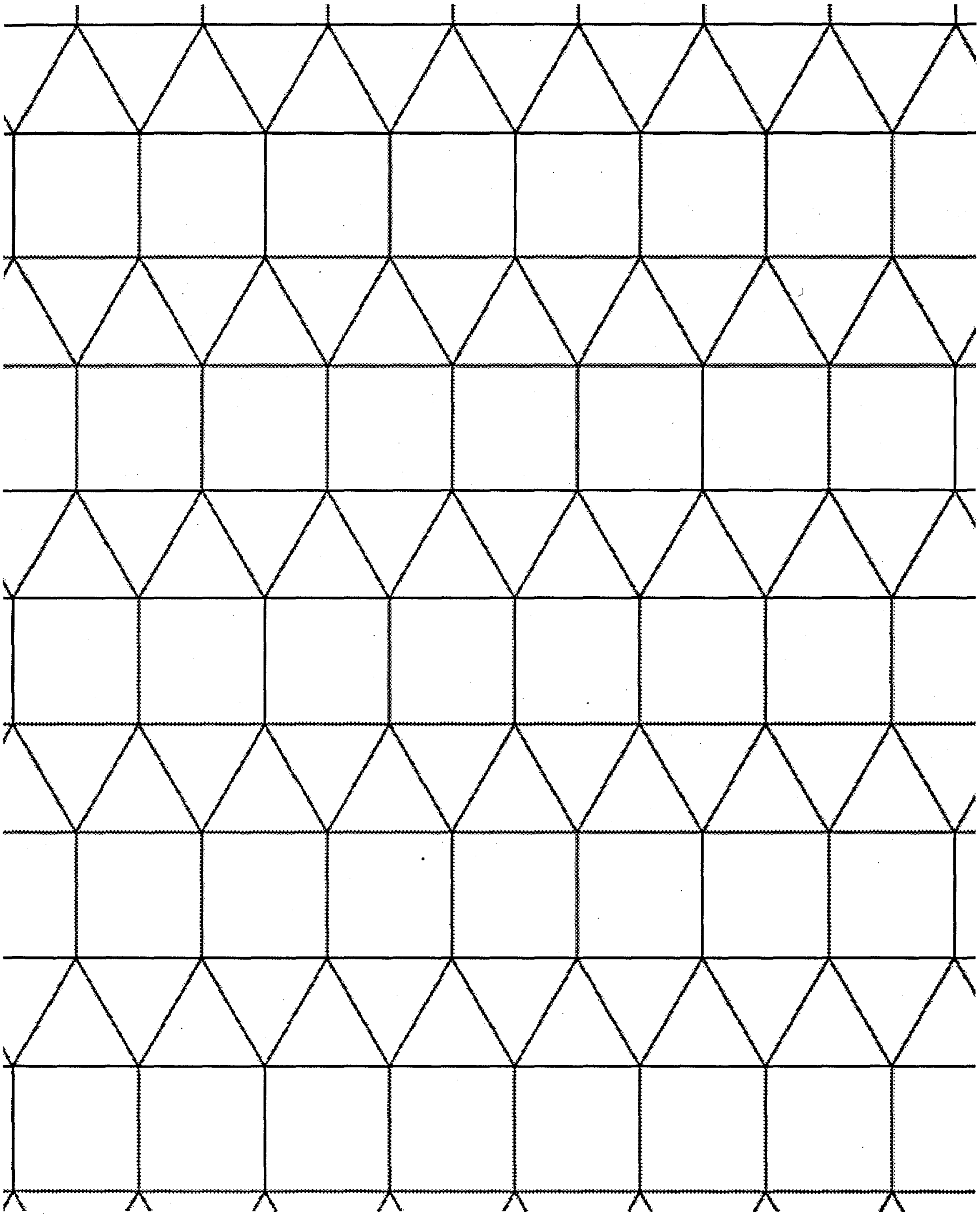

Semiregular Tessellation 3.3.4.3.4 (black large image)

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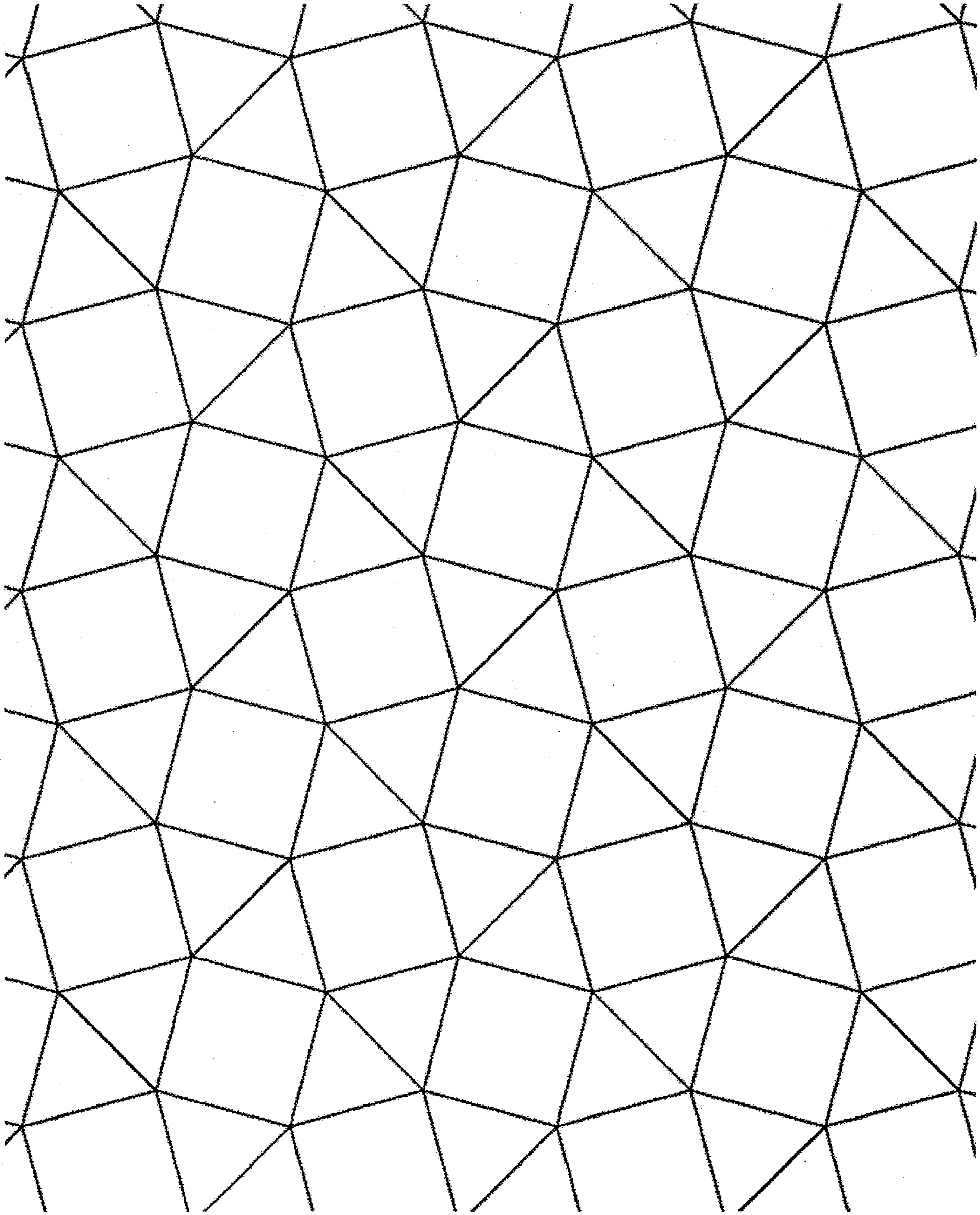
Instructions: Print this page and then close this window.

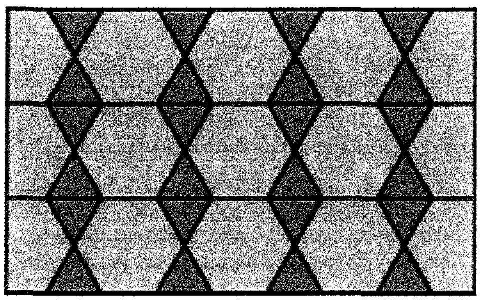


Demiregular Tessellations (2/2)

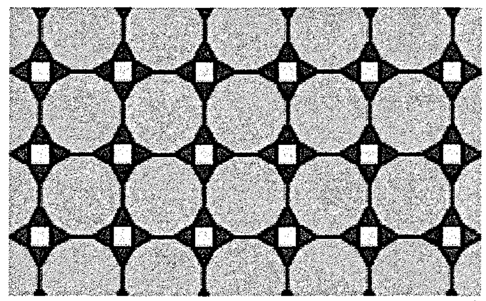
The Demiregular Tessellations

tessellations of regular polygons and that have exactly two or three different polygon arrangements

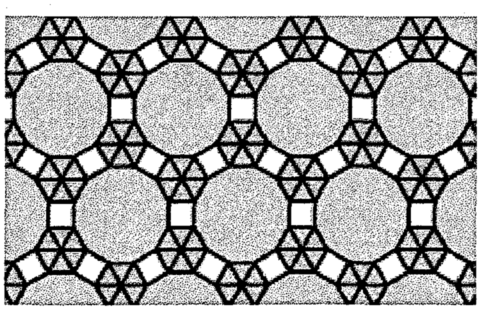
those with two different polygon arrangements:



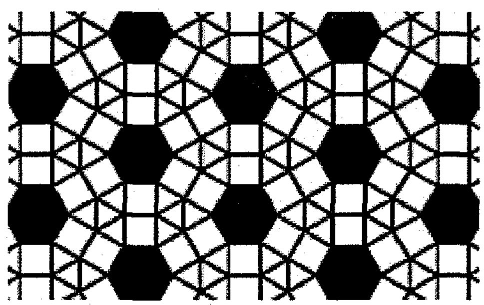
3.3.6.6 / 3.6.3.6



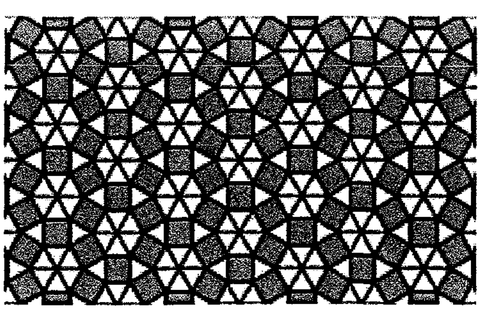
3.12.12 / 3.4.3.12



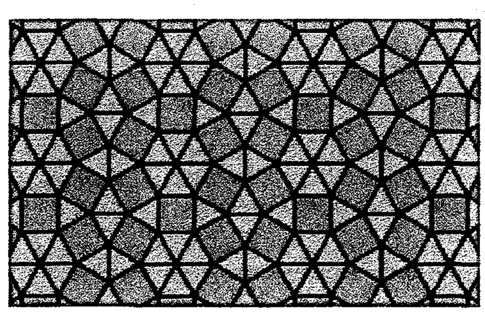
3.3.3.3.3.3 / 3.3.4.12



3.3.3.4.4 / 3.4.6.4

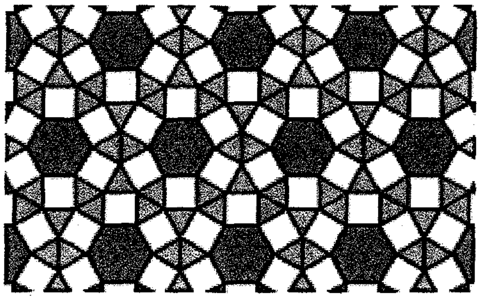


3.3.3.3.3.3 / 3.3.4.3.4 #1

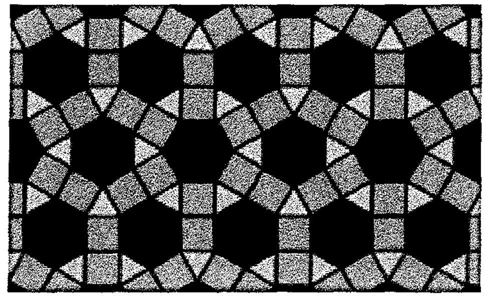


3.3.3.3.3.3 / 3.3.4.3.4 #2

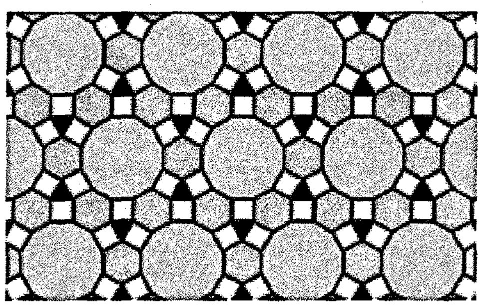
note that although the above two tessellations use the same polygon arrangements, they differ in their overall structure



3.3.4.3.4 / 3.4.6.4



3.4.6.4 / 3.4.4.6



3.4.6.4 / 4.6.12

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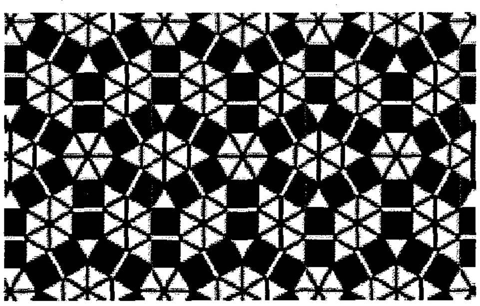
those with three different polygon arrangements:



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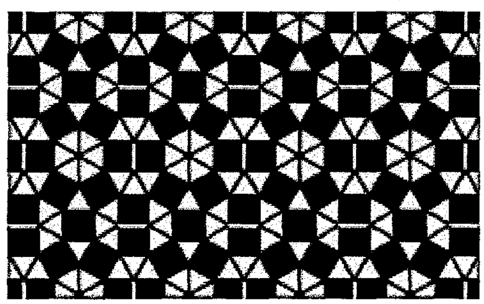
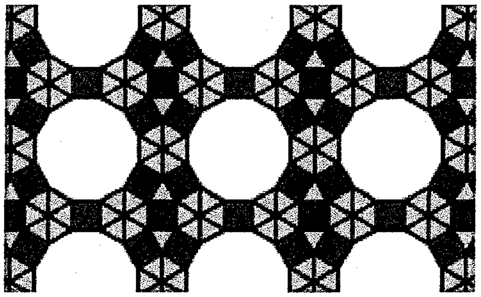
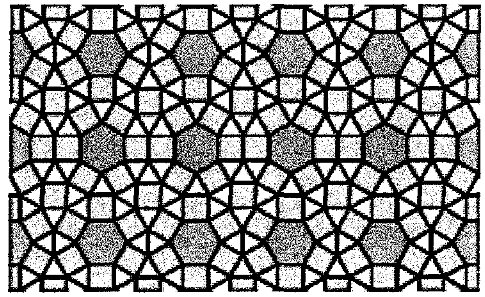


3.3.3.3.3.3 / 3.3.3.4.4 / 3.3.4.3.4 #2

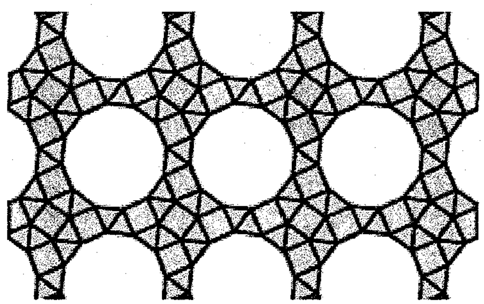
note that although the above two tessellations use the same polygon arrangements, they differ in their overall structure



3.3.3.3.3.3 / 3.3.4.12 / 3.3.4.3.4



3.3.3.4.4 / 3.3.4.3.4 / 3.4.6.4



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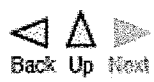
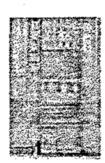


A real example of a demiregular tessellation:



To browse full-page templates of the demiregular tessellations that are ready to be printed, proceed to the templates page:

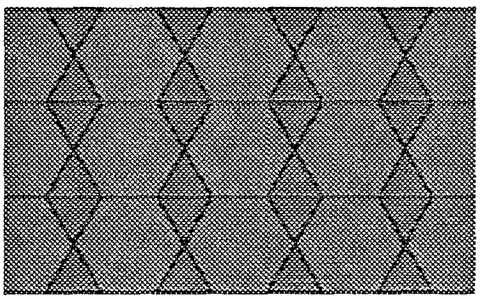
top of the page

Demiregular Tessellations (2/2)

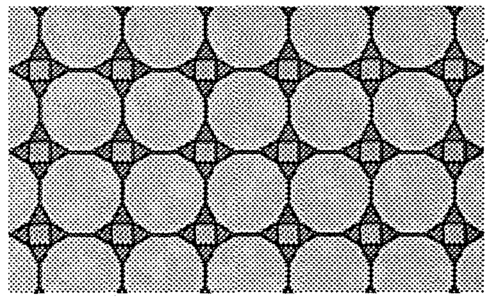
The Demiregular Tessellations

tessellations of regular polygons and that have exactly two or three different polygon arrangements

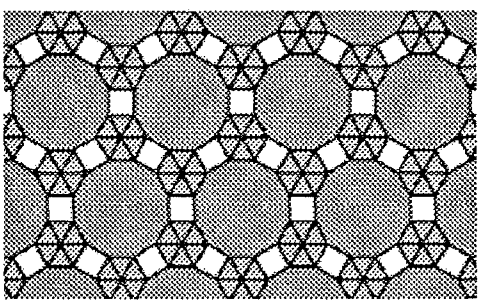
those with two different polygon arrangements:



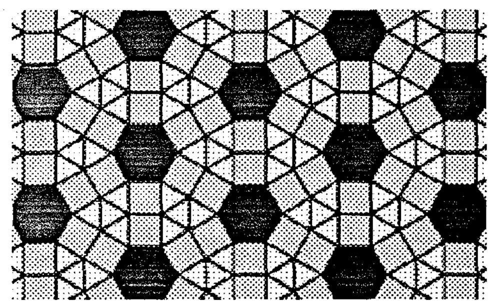
3.3.6.6 / 3.6.3.6



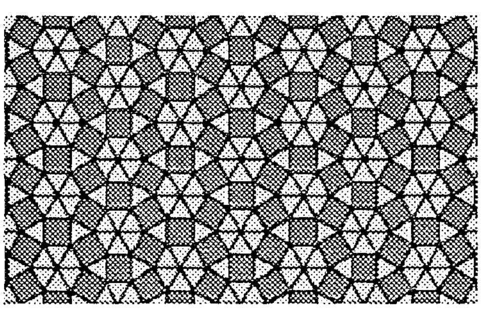
3.12.12 / 3.4.3.12



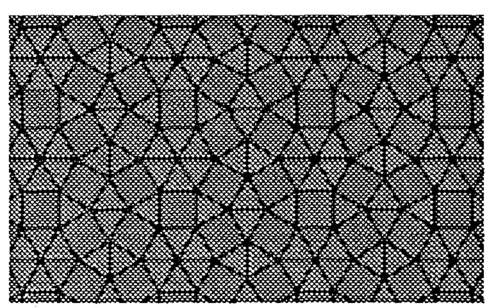
3.3.3.3.3.3 / 3.3.4.12



3.3.3.4.4 / 3.4.6.4

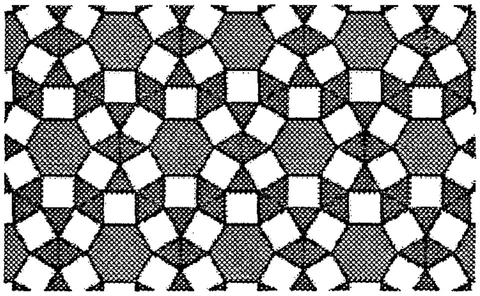


3.3.3.3.3.3 / 3.3.4.3.4 #1

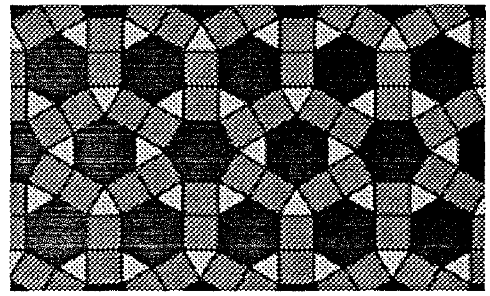


3.3.3.3.3.3 / 3.3.4.3.4 #2

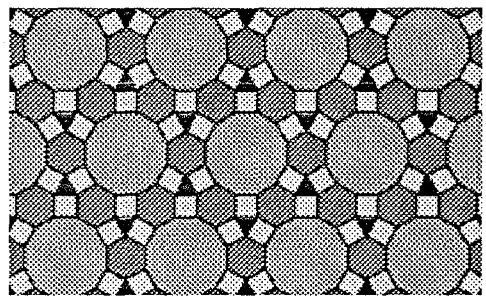
note that although the above two tessellations use the same polygon arrangements, they differ in their overall structure



3.3.4.3.4 / 3.4.6.4



3.4.6.4 / 3.4.4.6



3.4.6.4 / 4.6.12

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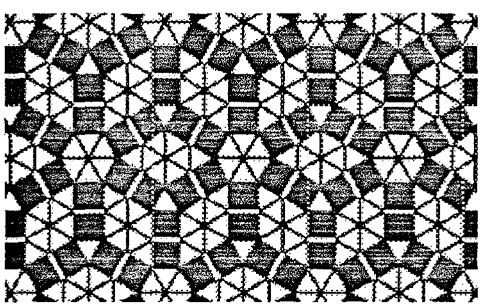
those with three different polygon arrangements:



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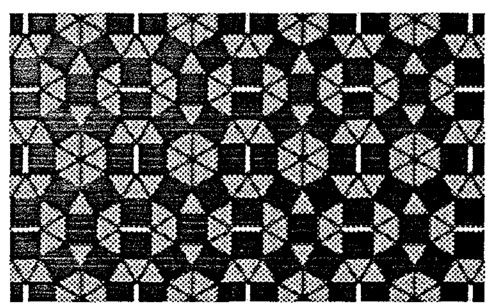
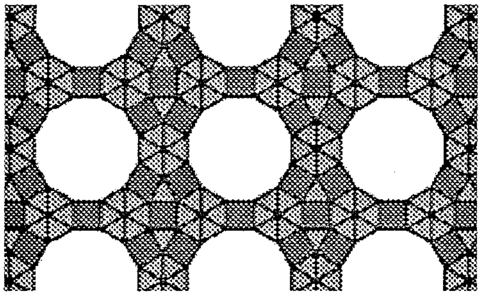
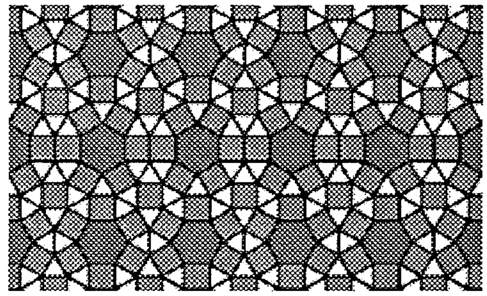


3.3.3.3.3.3 / 3.3.3.4.4 / 3.3.4.3.4 #2

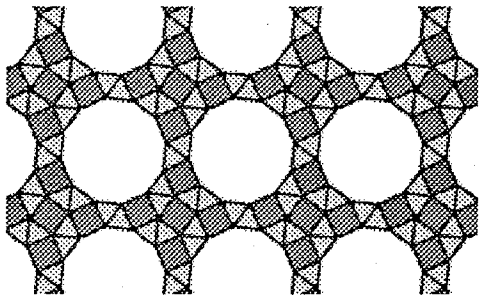
note that although the above two tessellations use the same polygon arrangements, they differ in their overall structure



3.3.3.3.3.3 / 3.3.4.12 / 3.3.4.3.4



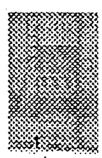
3.3.3.4.4 / 3.3.4.3.4 / 3.4.6.4



3.3.4.3.4 / 3.3.4.12 / 3.4.3.12

Back Up Next

A real example of a demiregular tessellation:



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Demiregular Tessellation 3.6.3.6 / 3.3.6.6 (black large image)

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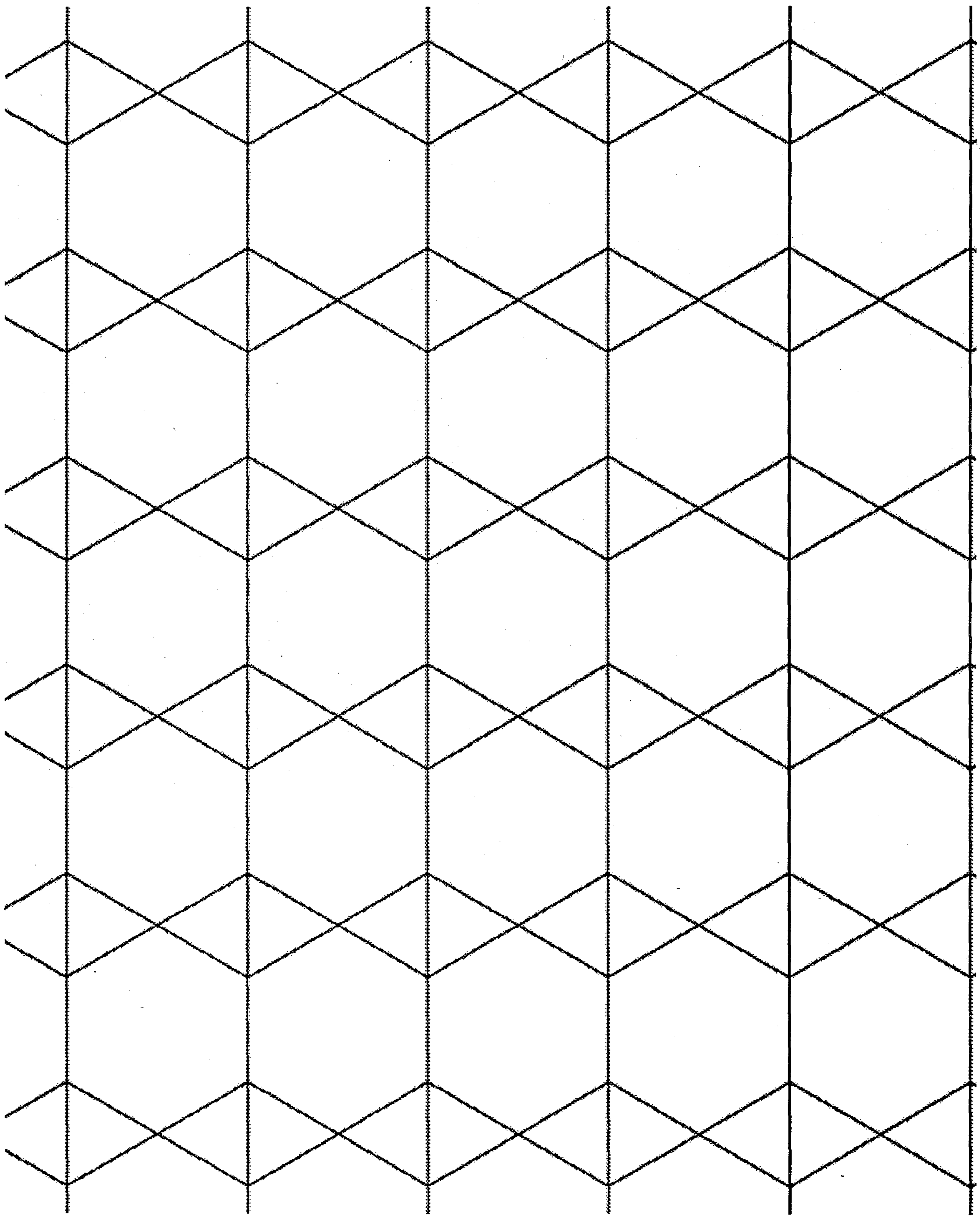

Demiregular Tessellation 3.12.12 / 3.4.3.12 (black large image)
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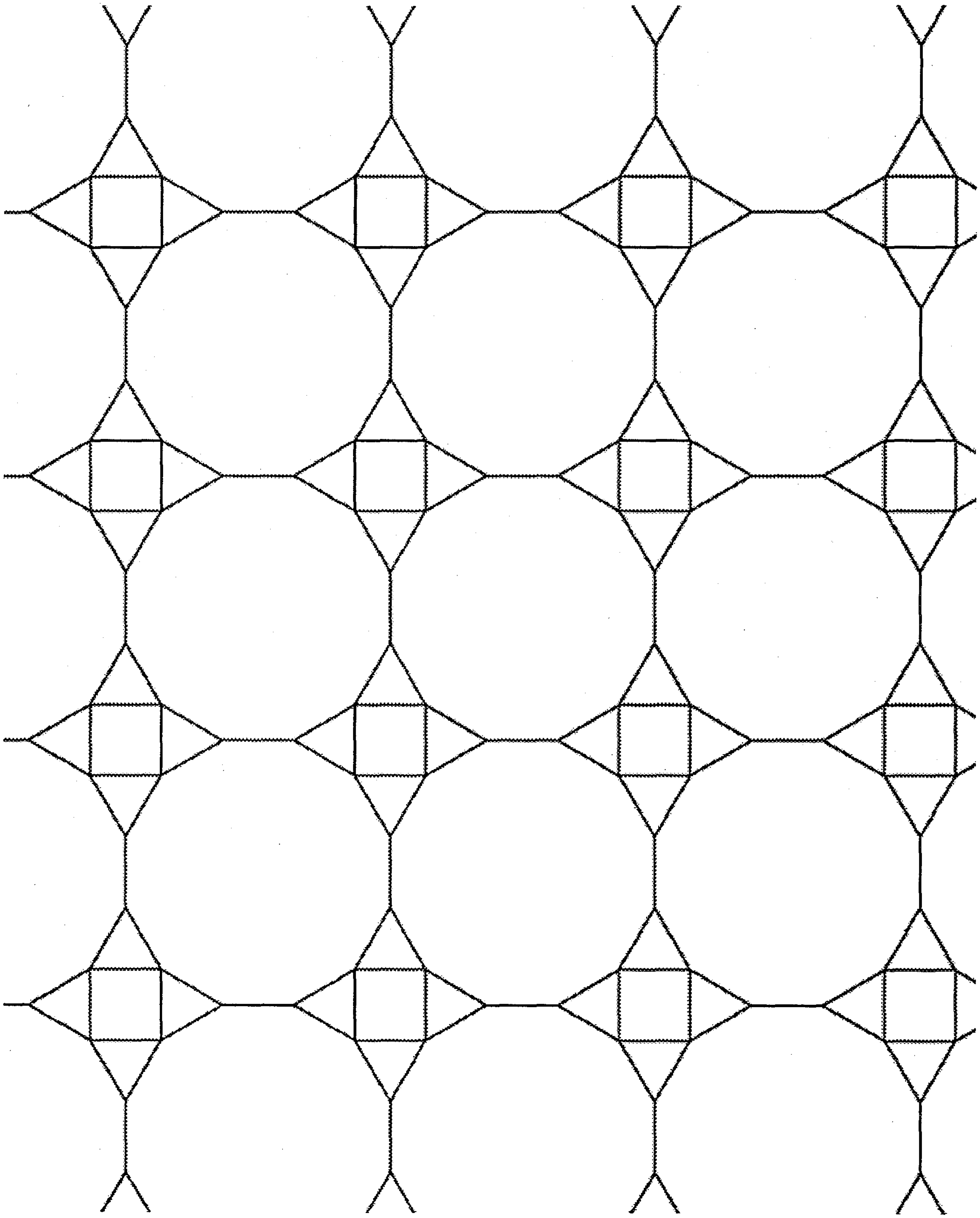

Demiregular Tessellation 3.3.3.3.3.3 / 3.3.4.12 (black large image)

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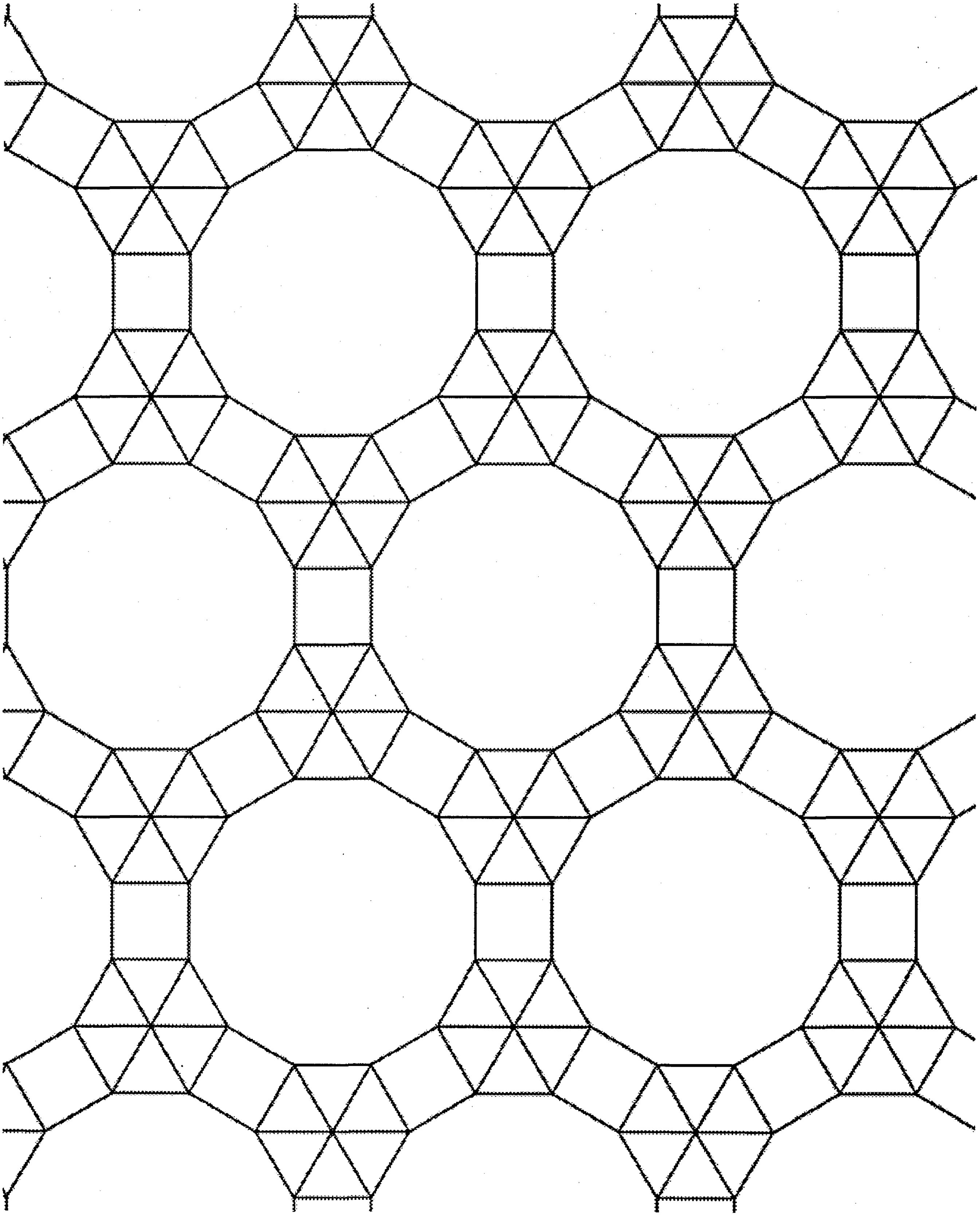

Demiregular Tessellation 3.3.3.4.4 / 3.4.6.4 (black medium image)

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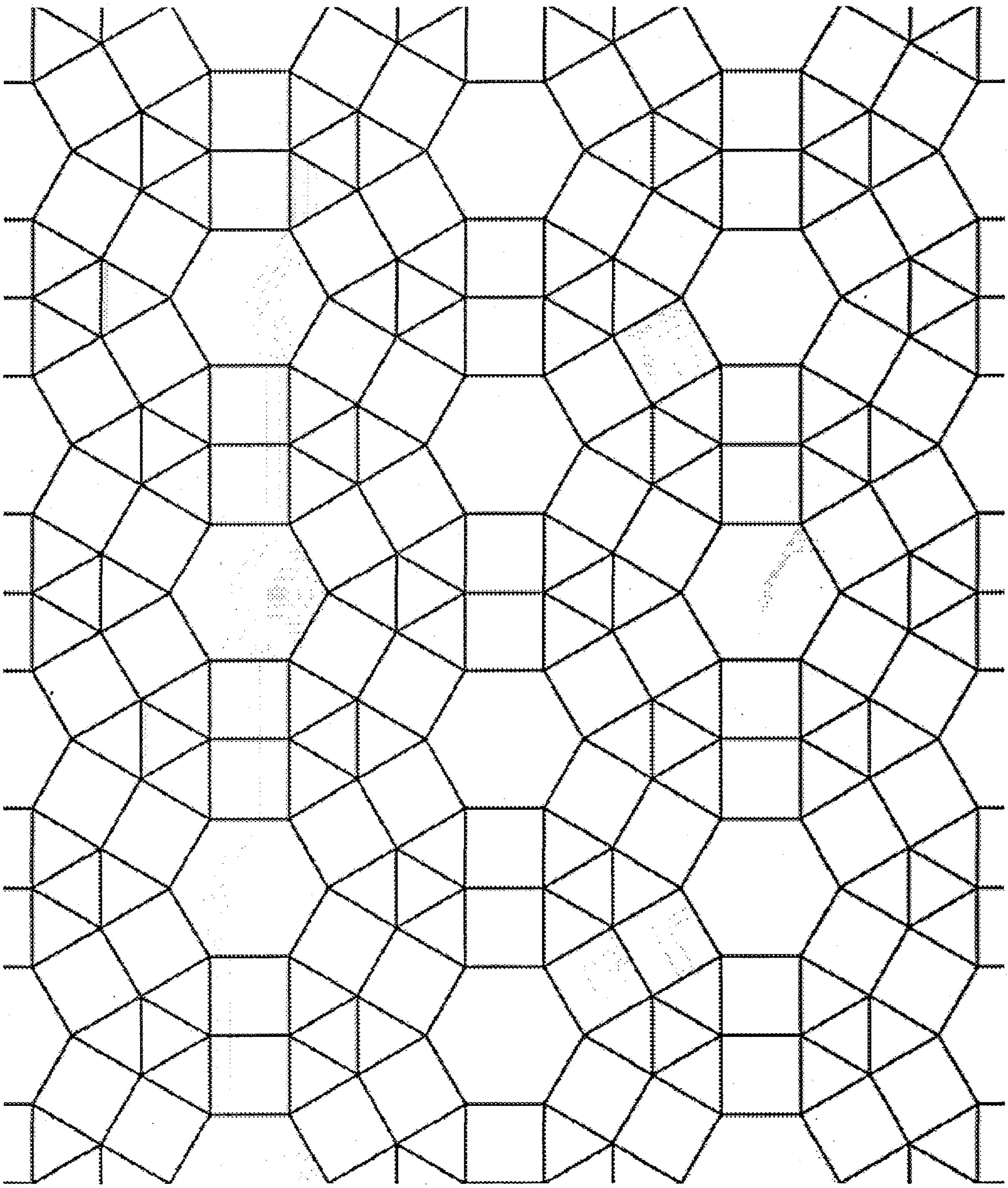

Demiregular Tessellation 3.3.3.3.3.3 / 3.3.4.3.4 #1 (black medium image)

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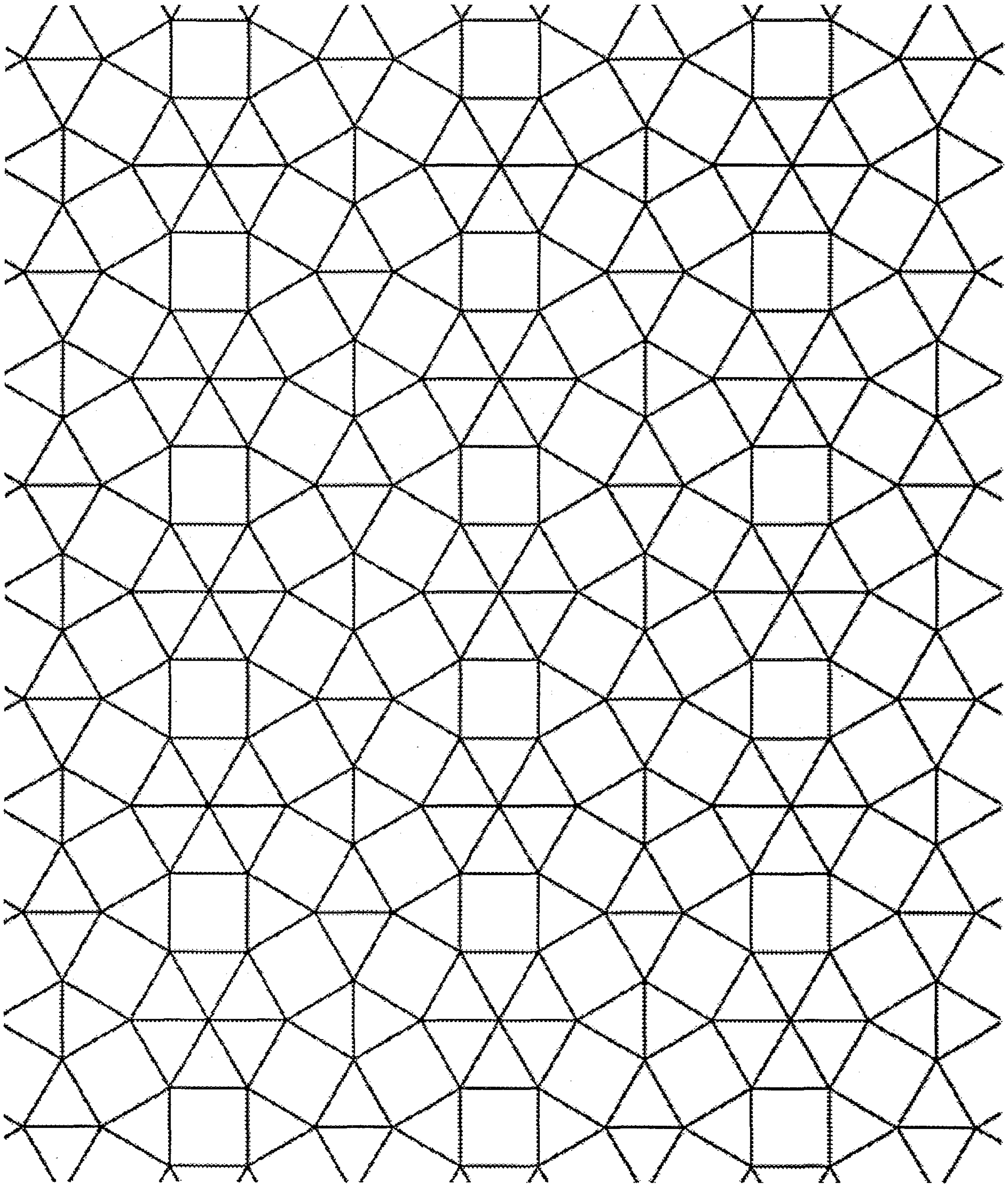

Demiregular Tessellation 3.3.3.3.3.3 / 3.3.4.3.4 #2 (black medium image)

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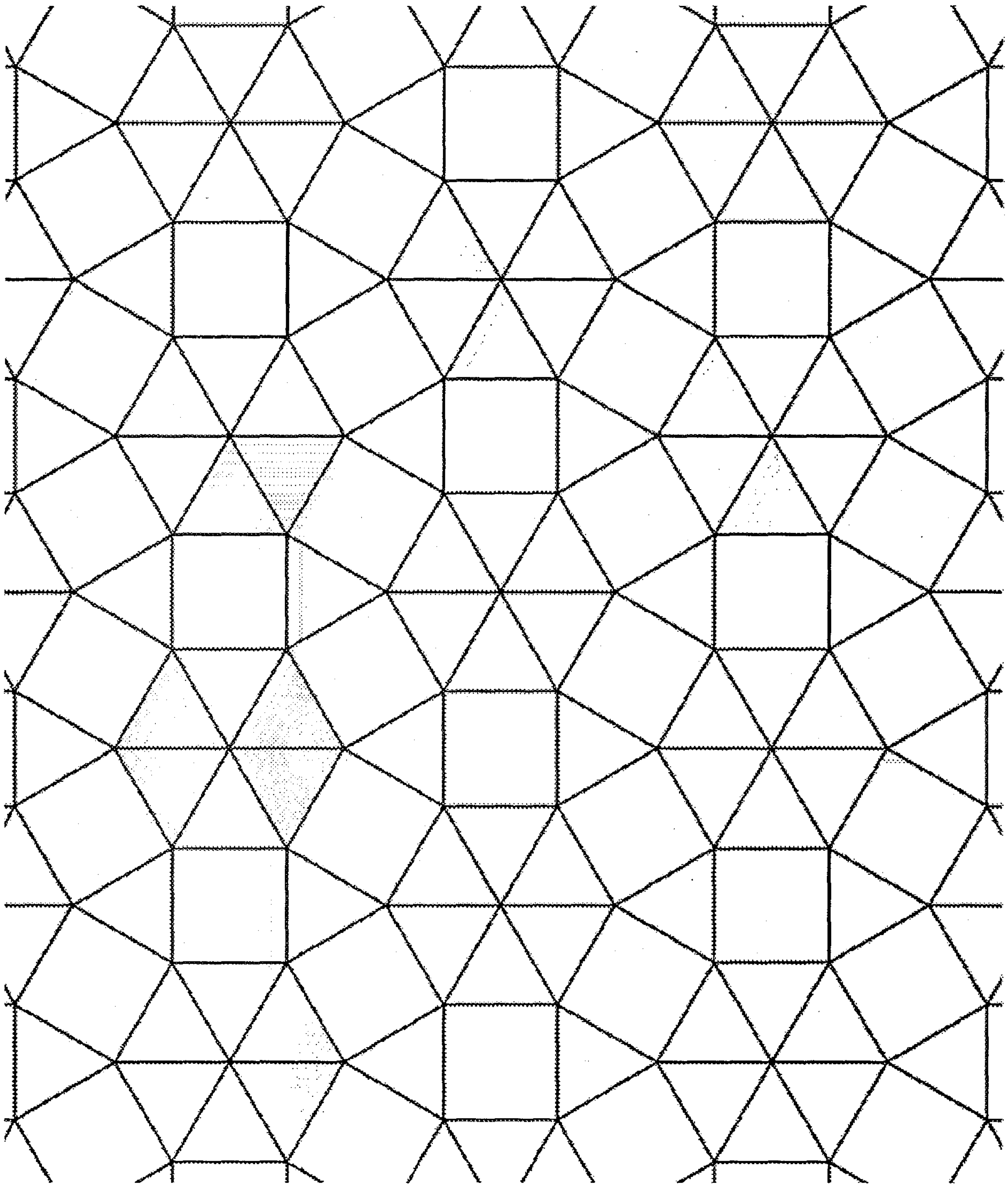
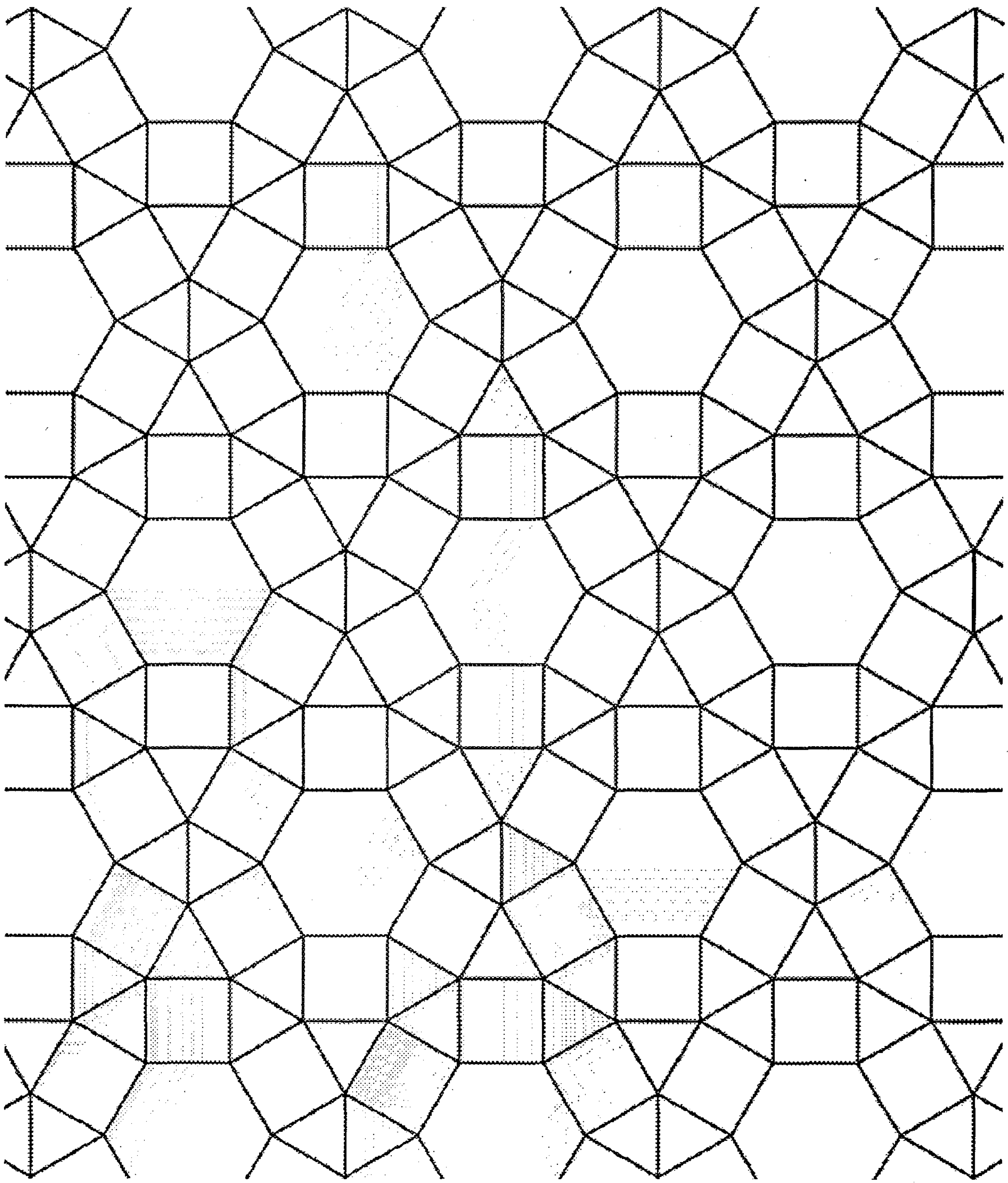


Demiregular Tessellation 3.3.4.3.4 / 3.4.6.4 (black medium image)
Totally Tessellated @ http://library.advanced.org/16661/
Instructions: Print this page and then close this window.



Demiregular Tessellation 3.4.6.4 / 3.4.4.6 (black medium image)

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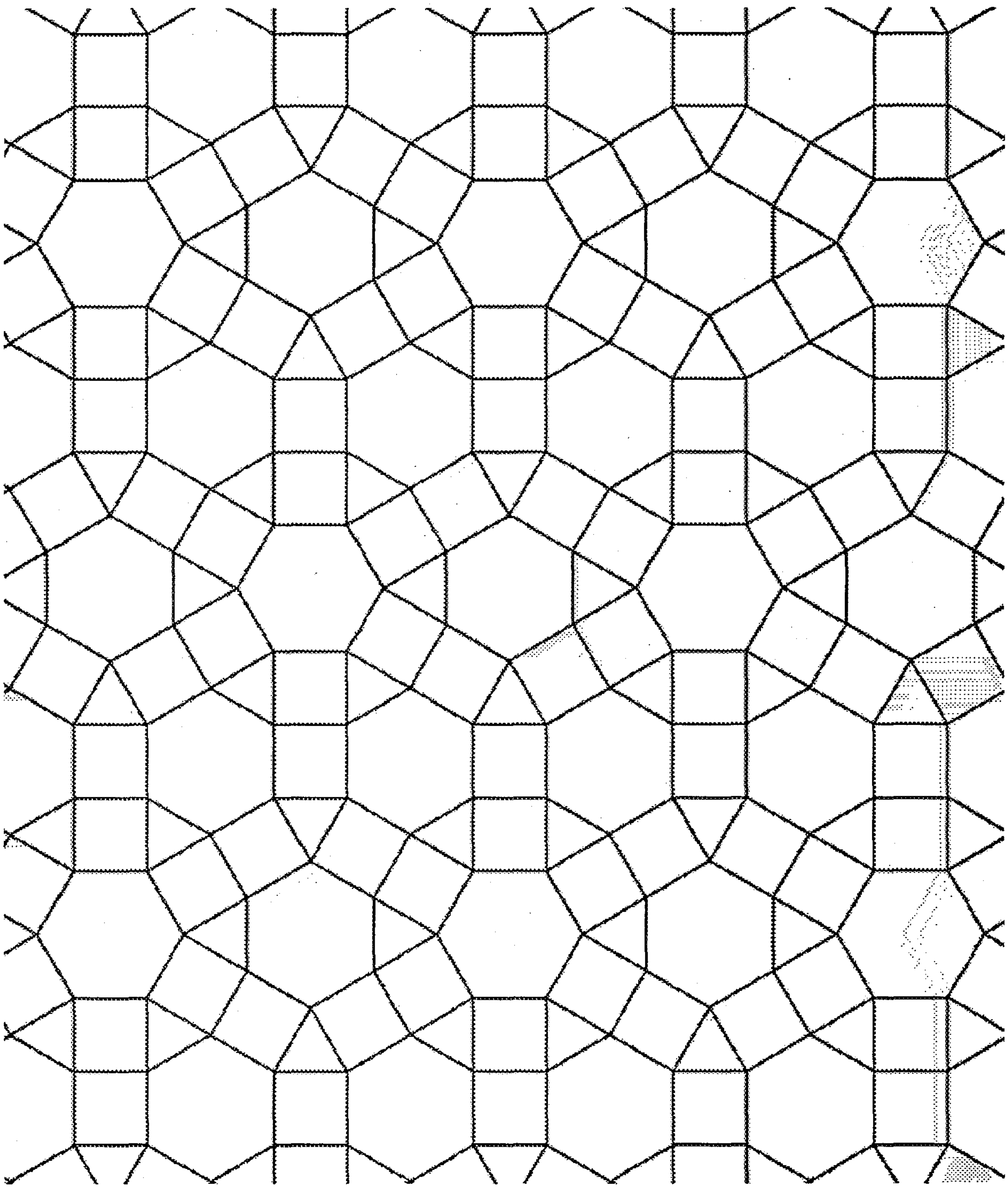

Demiregular Tessellation 3.4.6.4 / 4.6.12 (black large image)

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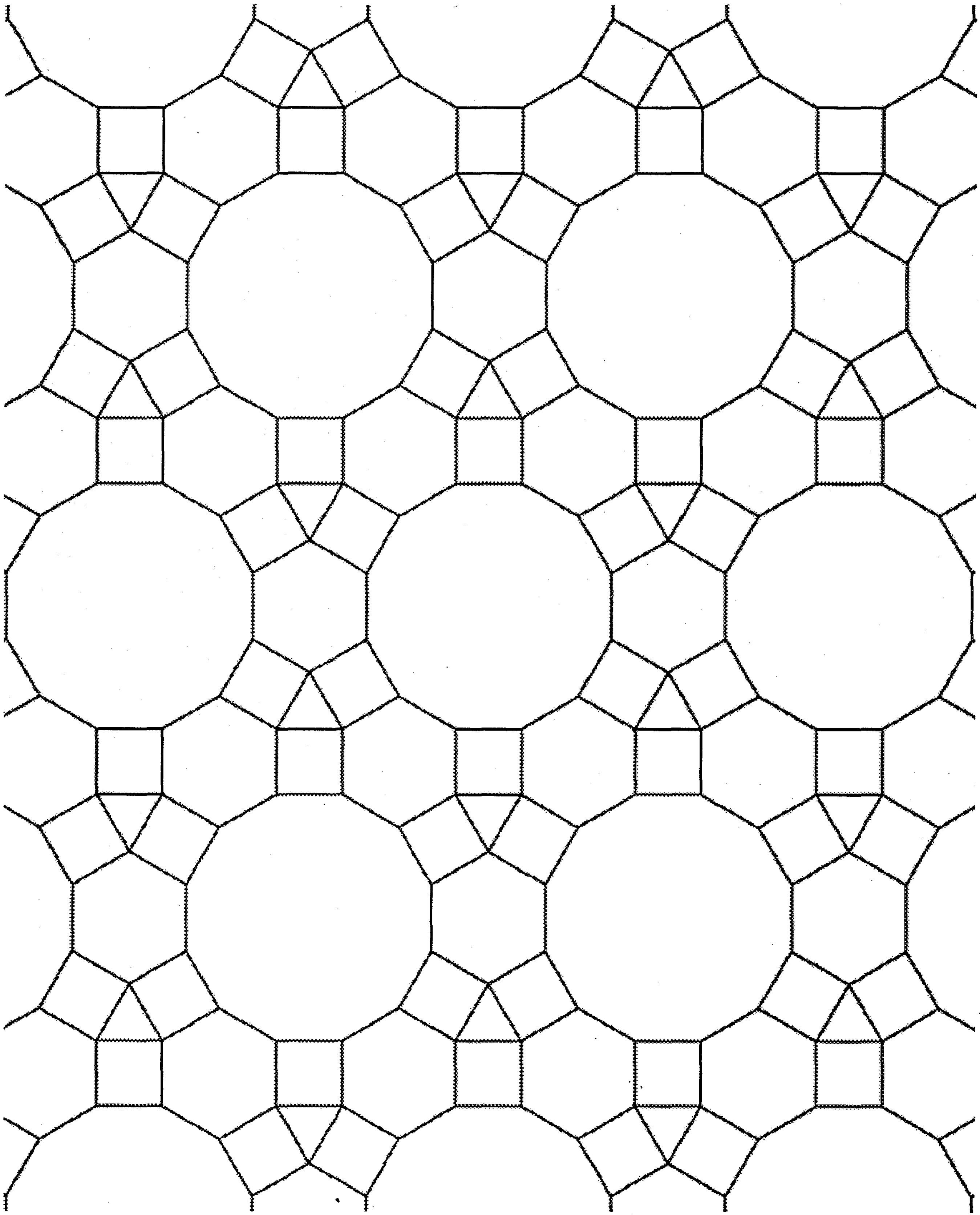

Demiregular Tessellation 3.3.3.3.3.3 / 3.3.3.4.4 / 3.3.4.3.4 #1 (black large image)

Totally Tessellated @ http://library.advanced.org/16661/

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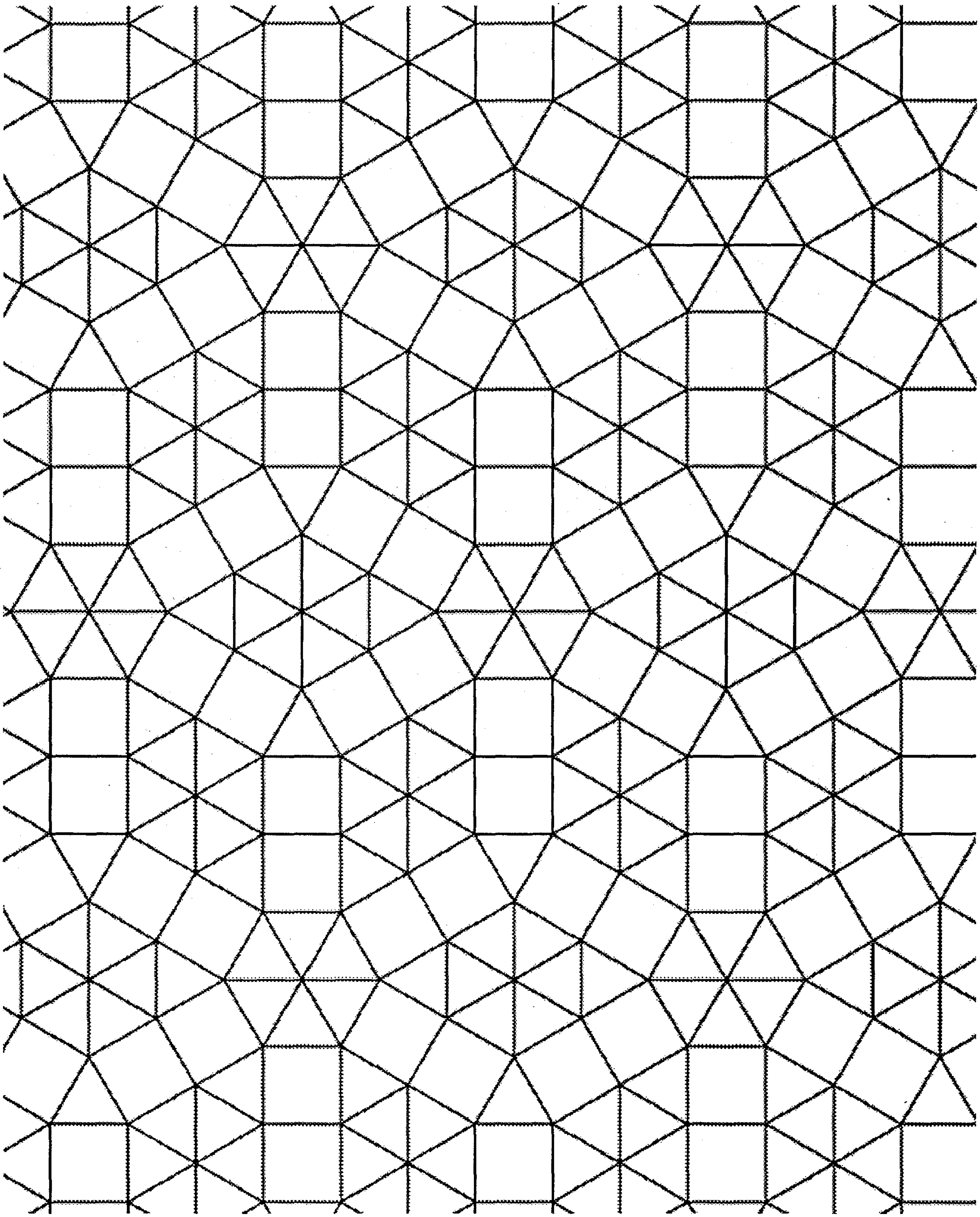

Demiregular Tessellation 3.3.3.3.3.3 / 3.3.3.4.4 / 3.3.4.3.4 #2 (black medium image)

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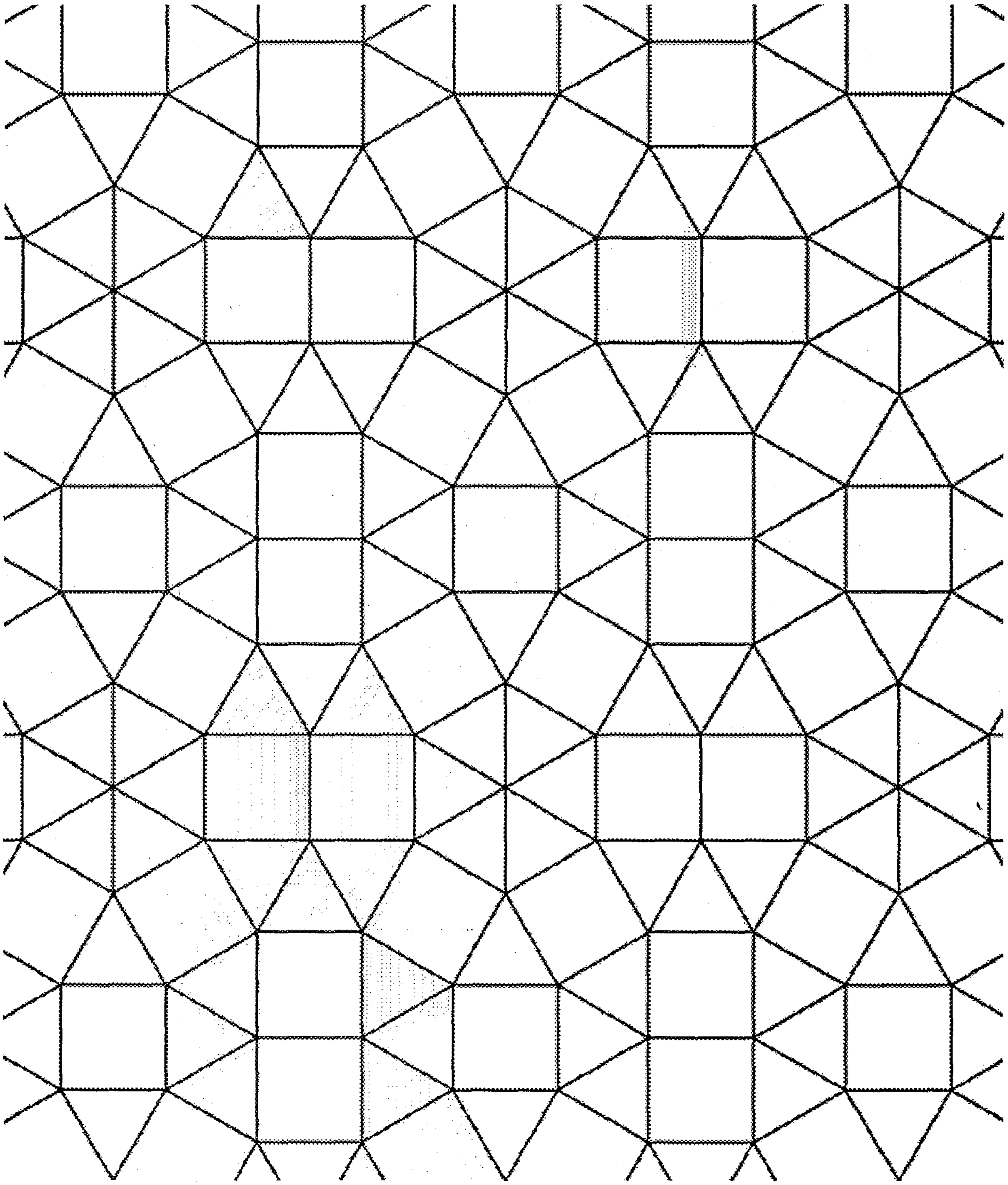

Demiregular Tessellation 3.3.3.3.3.3 / 3.3.4.12 / 3.3.4.3.4 (black medium image)

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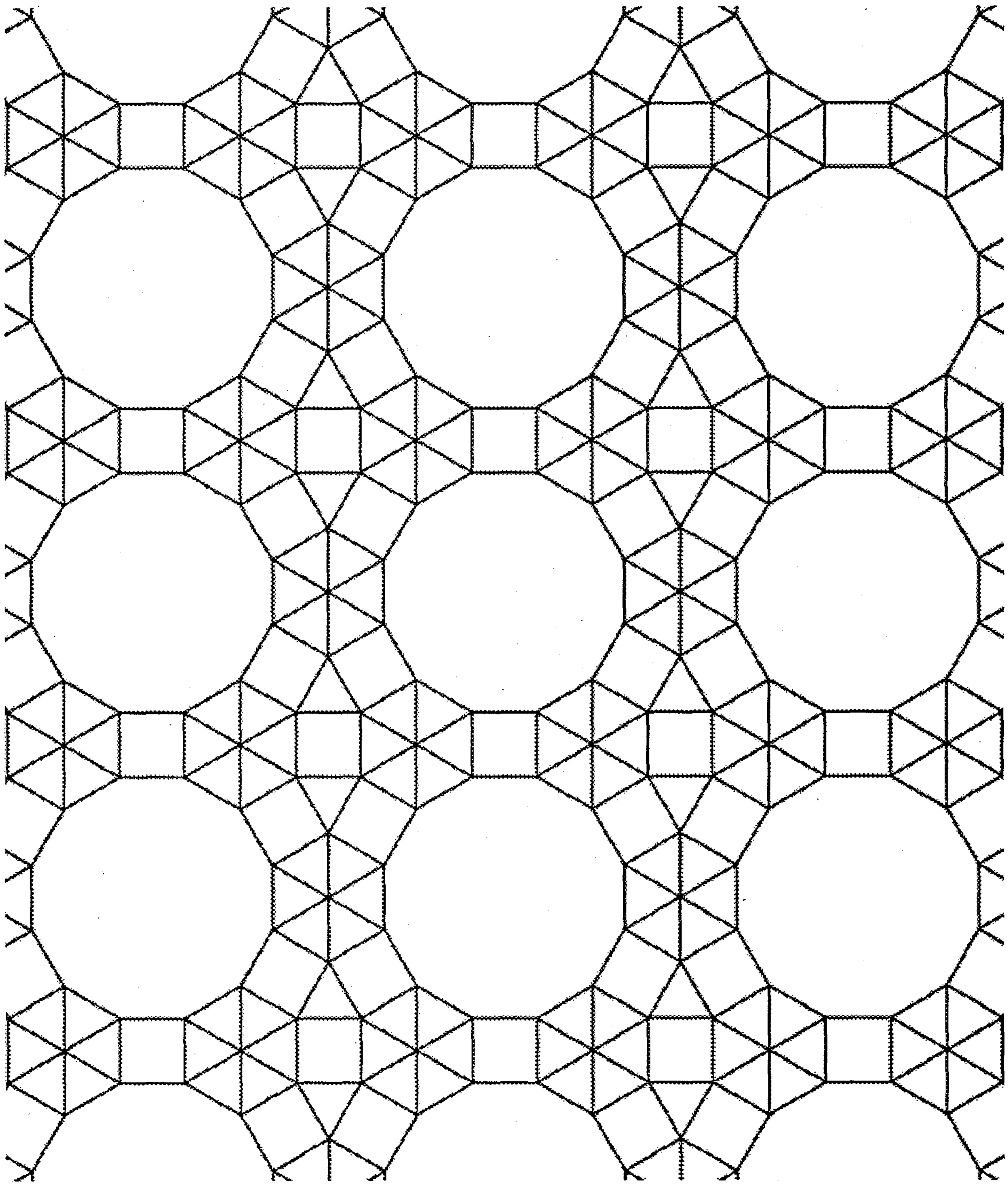

Demiregular Tessellation 3.3.3.4.4 / 3.3.4.3.4 / 3.4.6.4 (black medium image)

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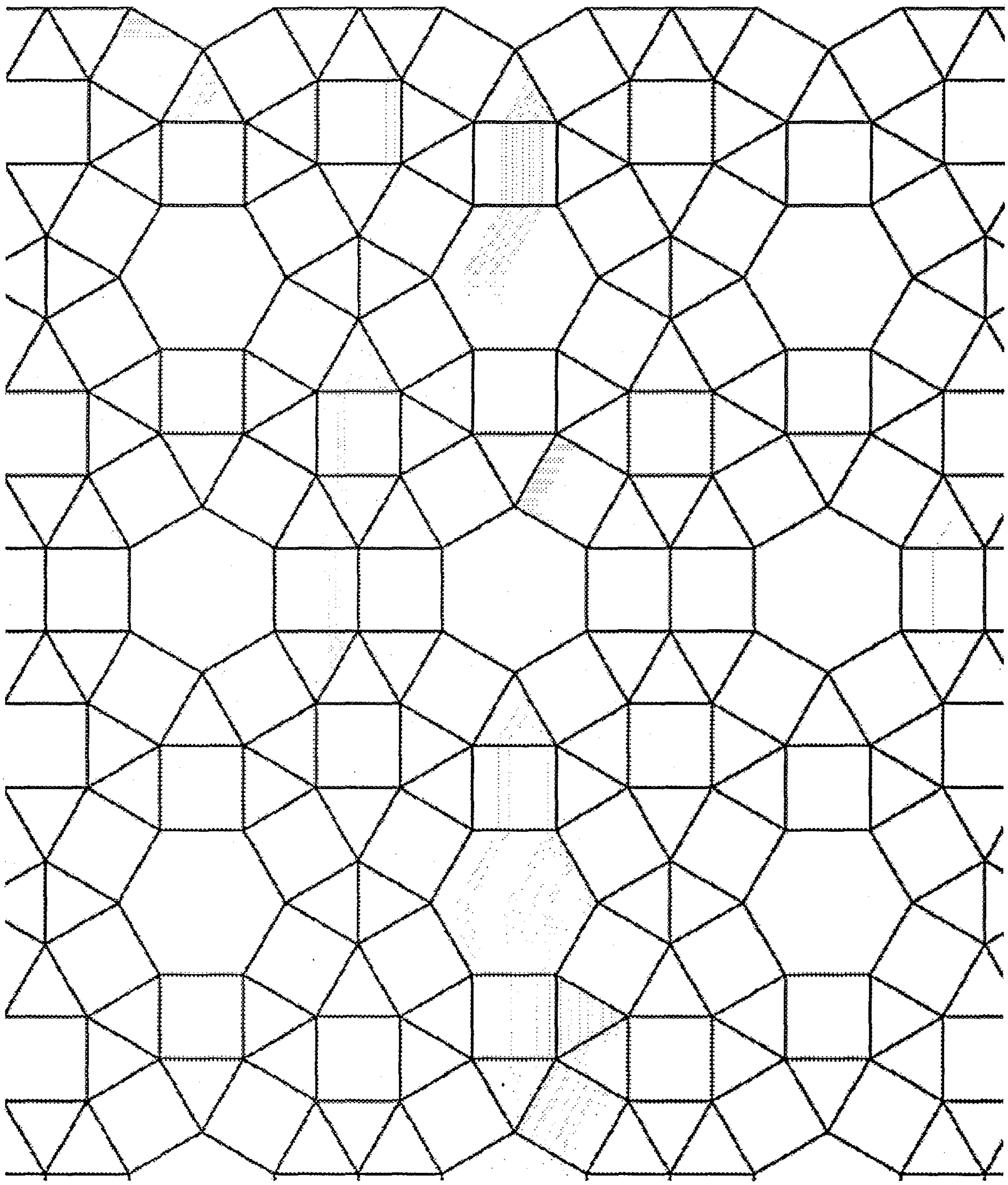

Demiregular Tessellation 3.3.4.3.4 / 3.3.4.12 / 3.4.3.12 (black medium image)

Totally Tessellated @ http://library.advanced.org/16661/

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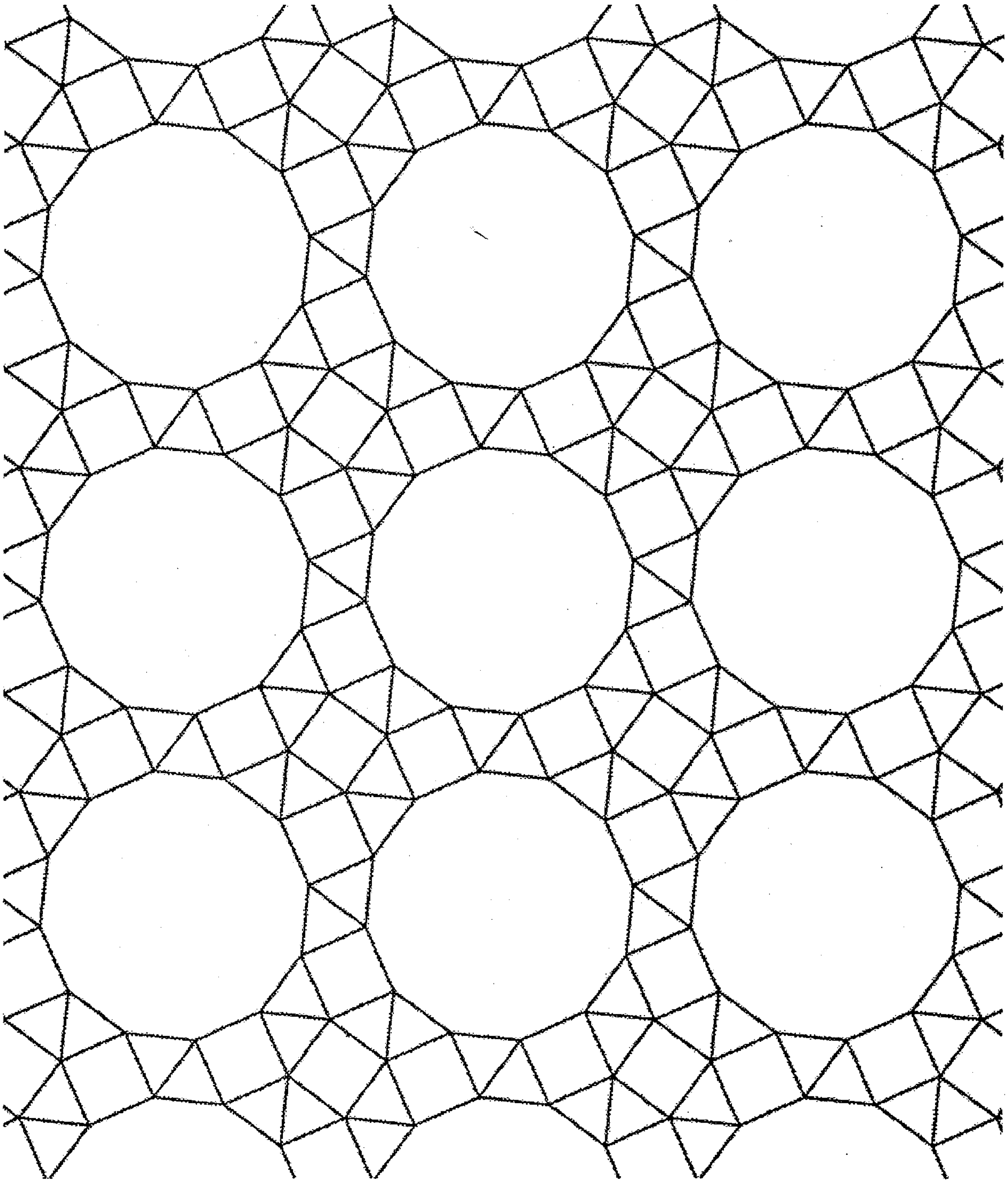

Extra Tessellation #1 (medium image)

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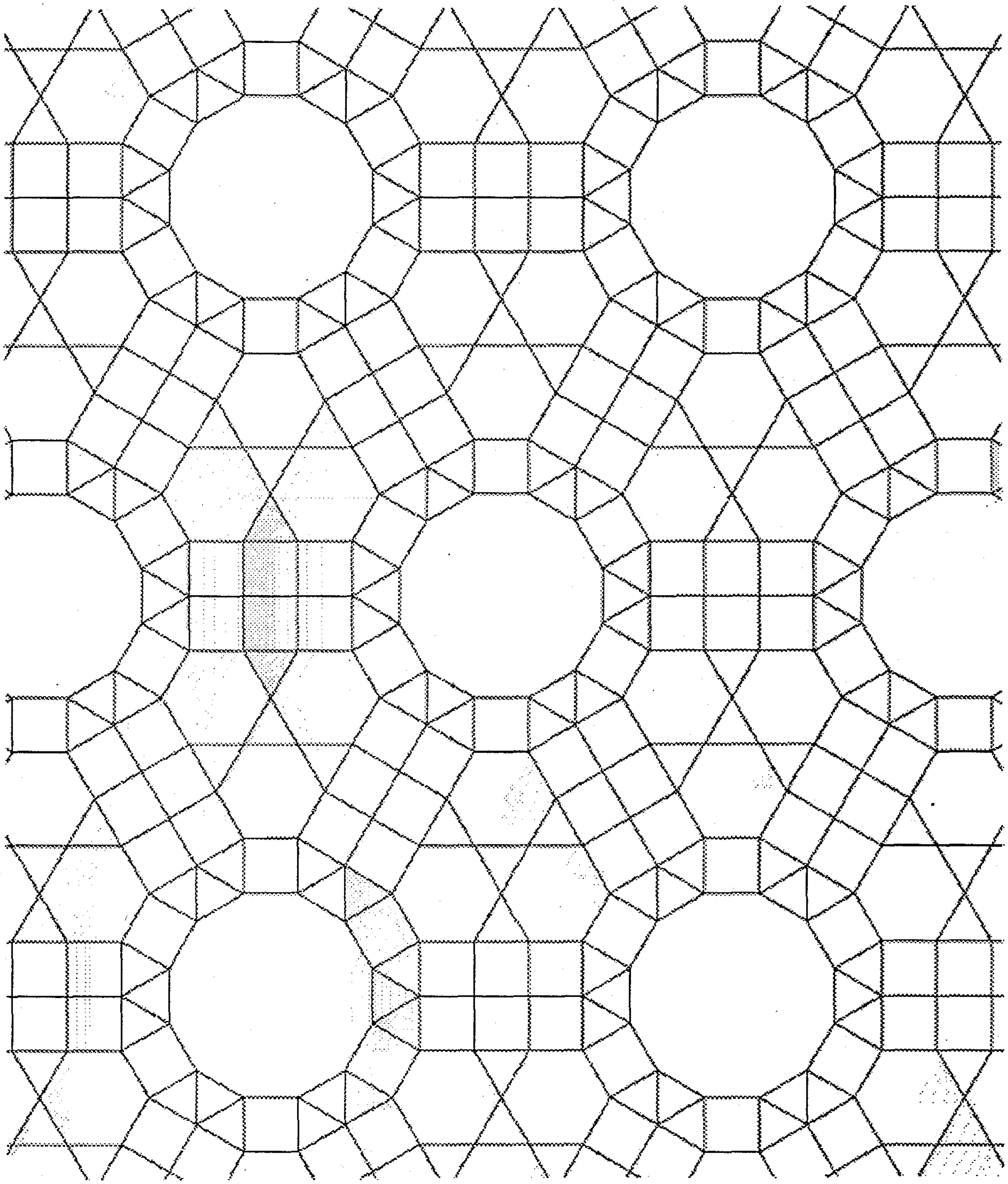

Extra Tessellation #2 (medium image)

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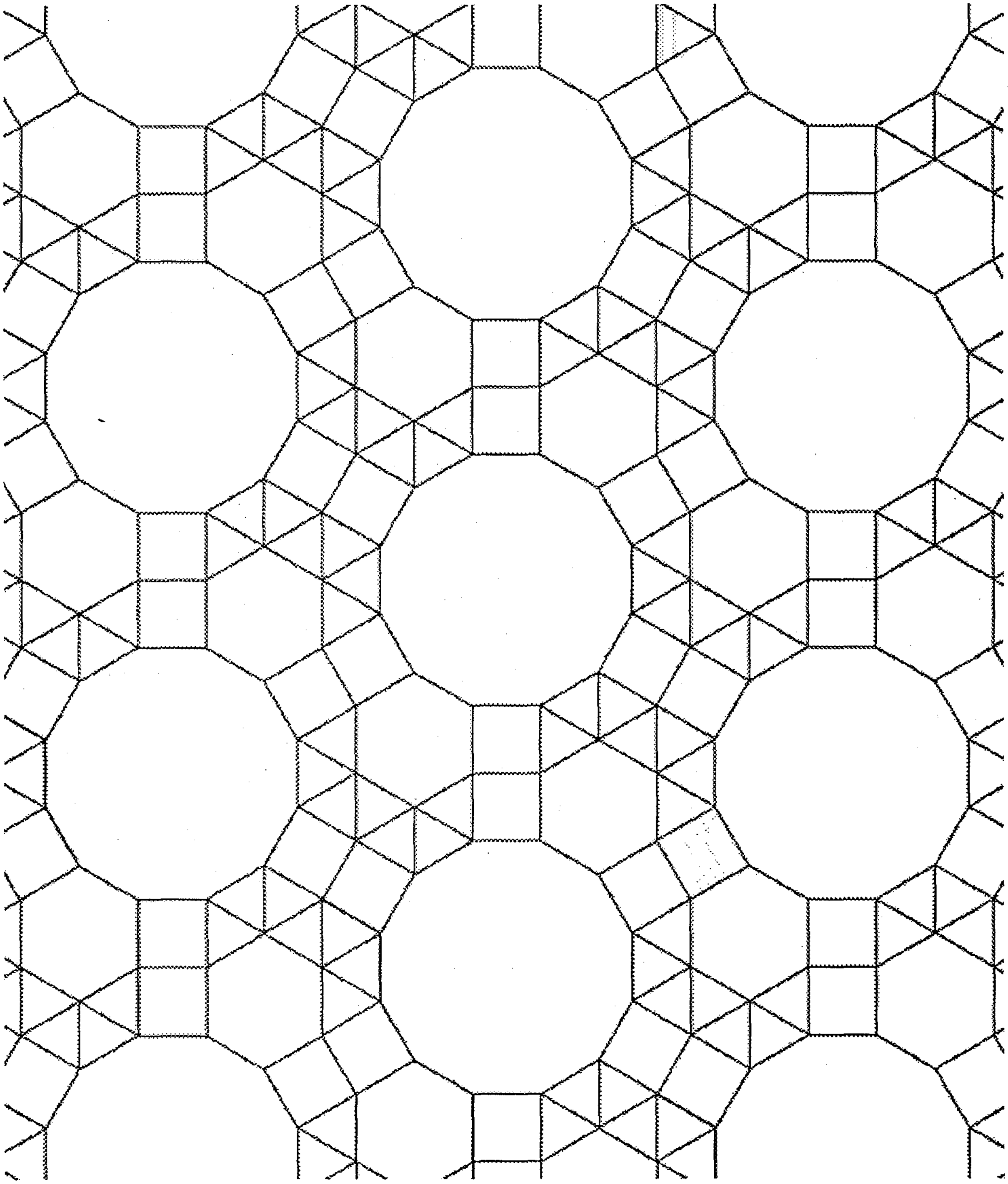

Extra Tessellation #3 (medium image)
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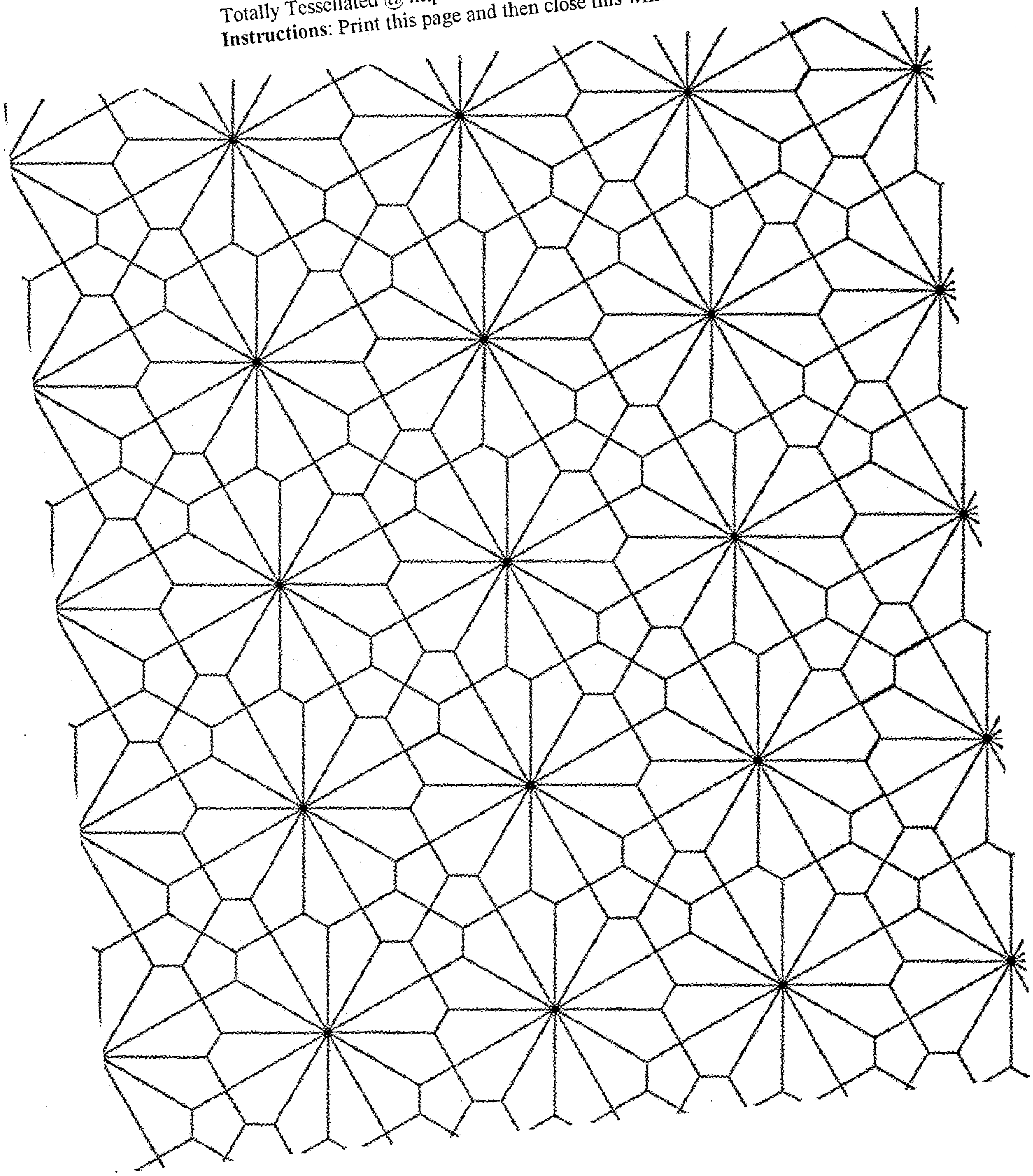

Extra Tessellation #4 (medium image)

Totally Tessellated @ http://library.advanced.org/16661/

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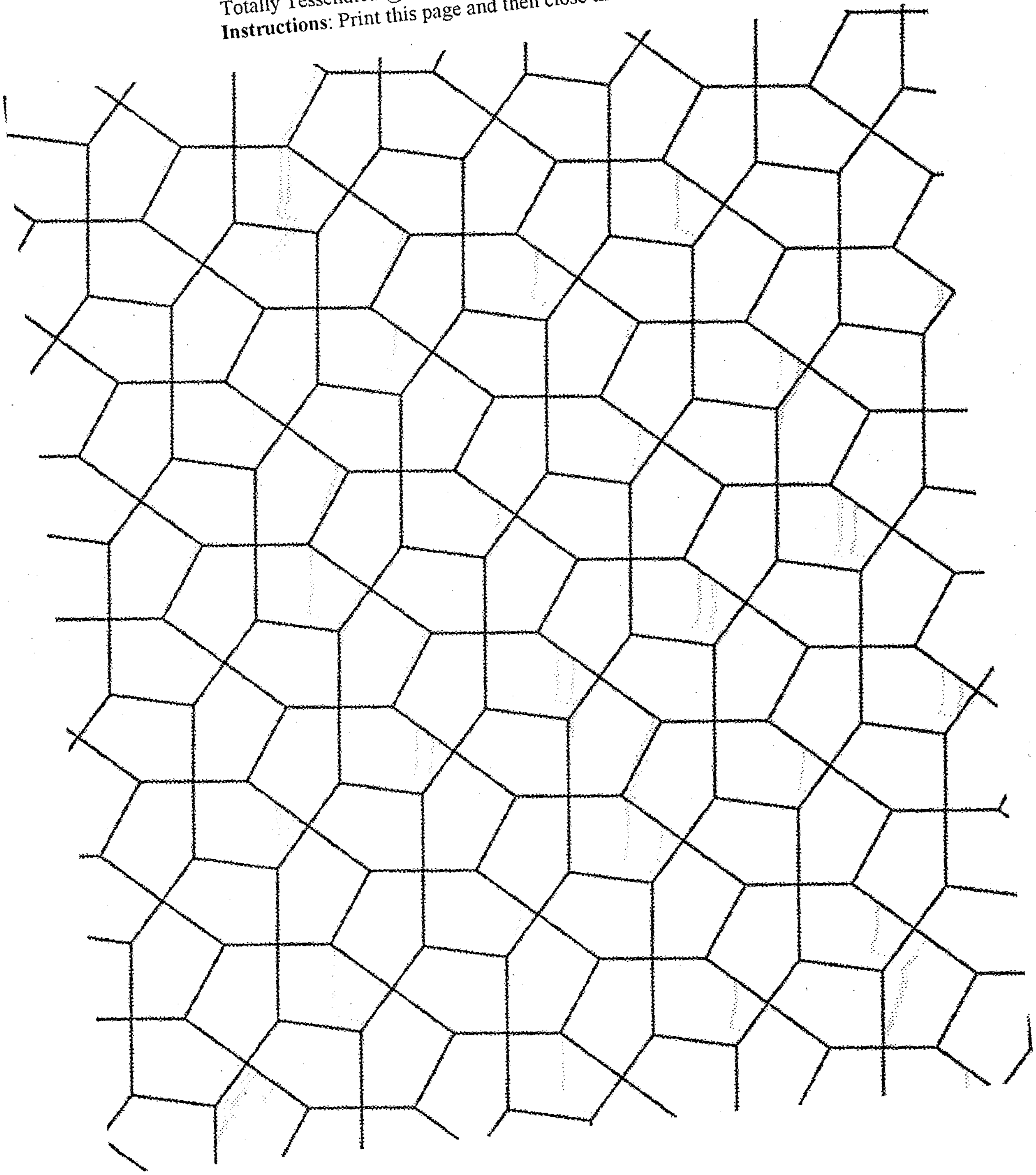
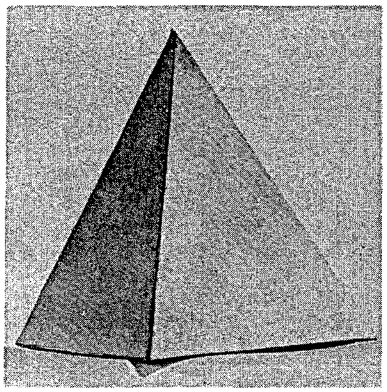
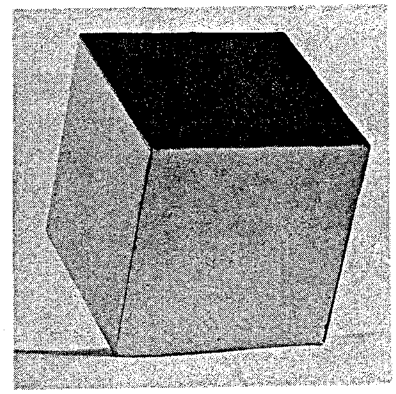


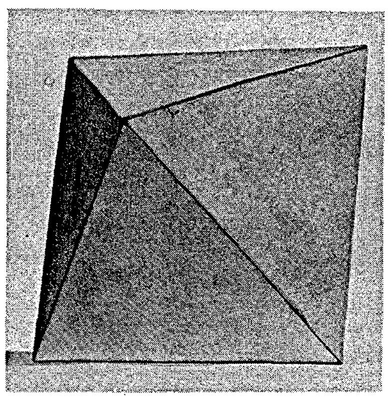
PLATO'S FIVE REGULAR POLYHEDRA



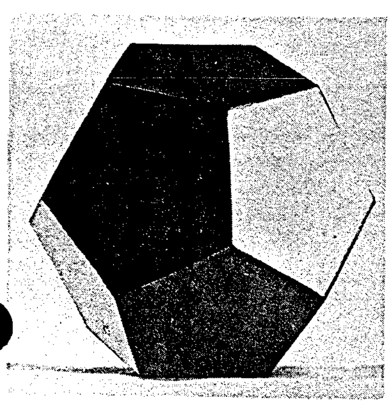
TETRAHEDRON



HEXAHEDRON (CUBE)



OCTAHEDRON



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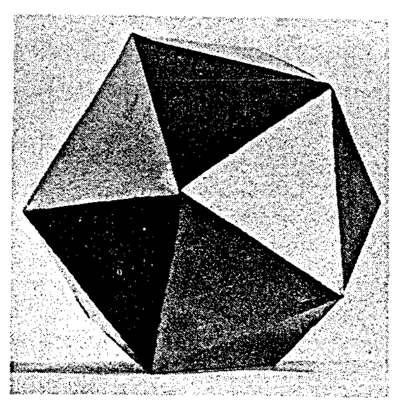
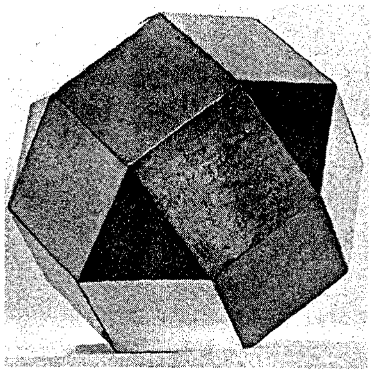
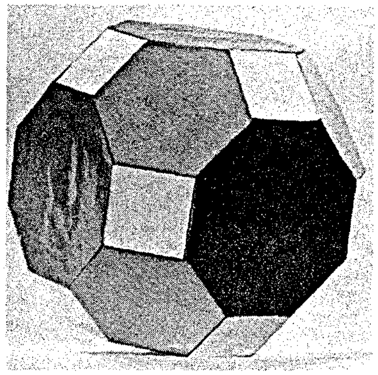


ICOSAHEDRON

ARCHIMEDEAN POLYHEDRA



SMALL RHOMBICUBOCTHEDRON



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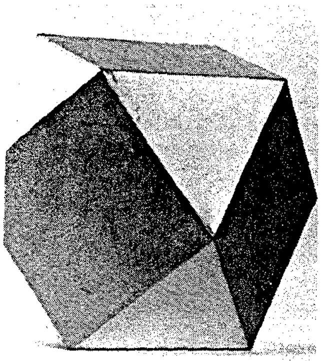
(TRUNCATED CUBOCTAHEDRON)



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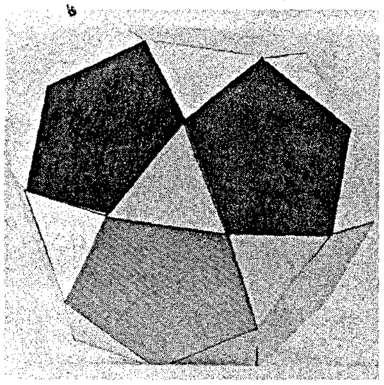
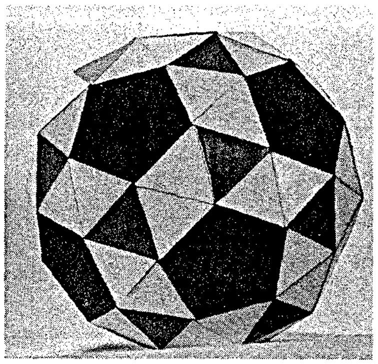


ICOSIDODECAHEDRON



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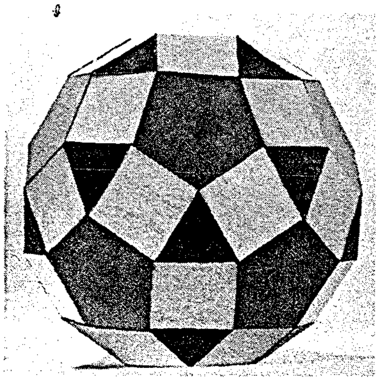

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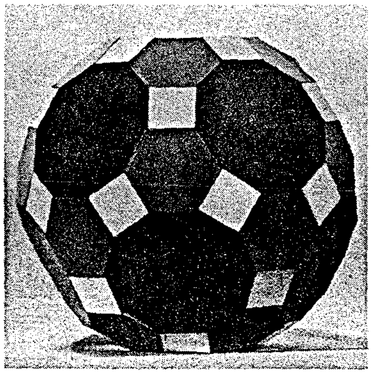

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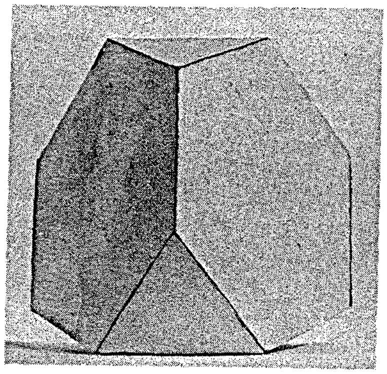
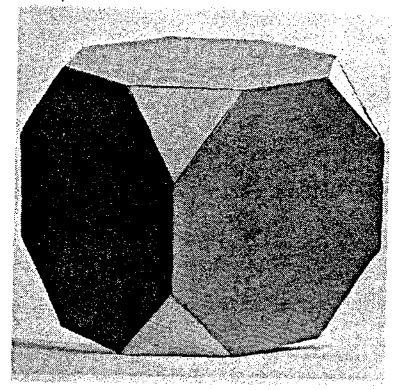
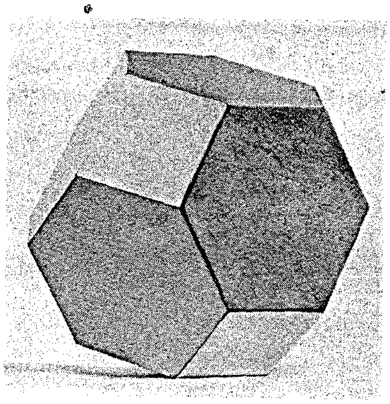
ARCHIMEDEAN POLYHEDRA



TRUNCATED
TETRAHEDRON



TRUNCATED CUBE



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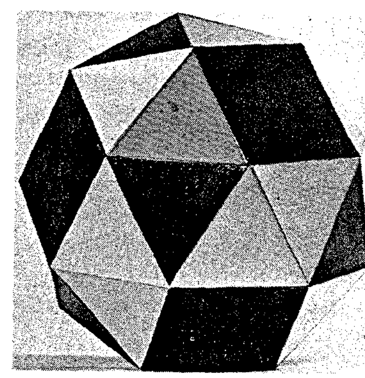

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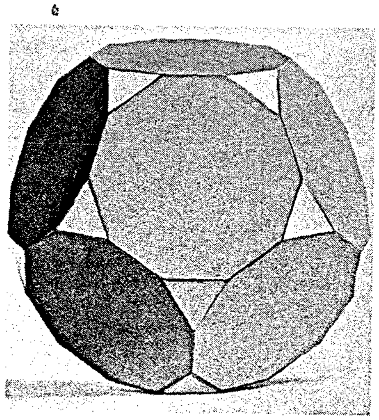

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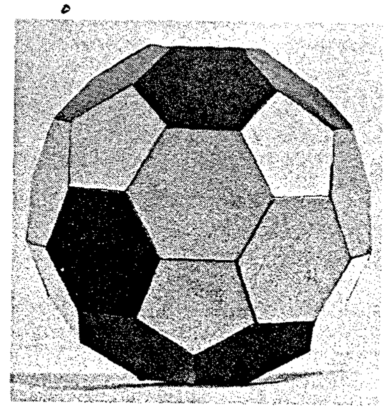
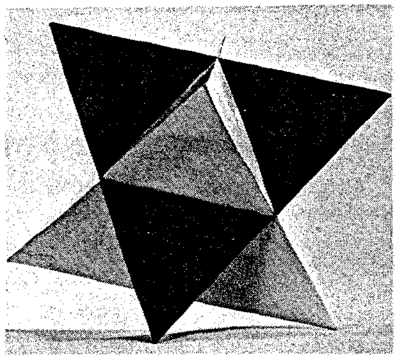
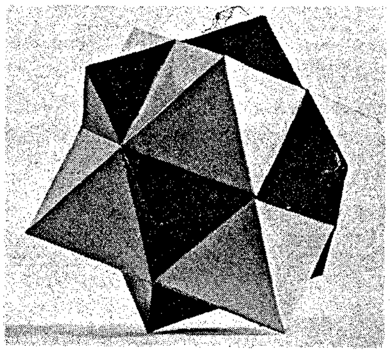


TRUNCATED
ICOSAHEDRON

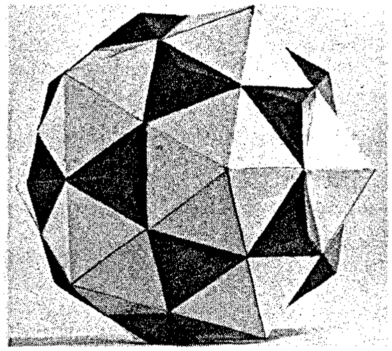
MISCELLANEOUS POLYHEDRA



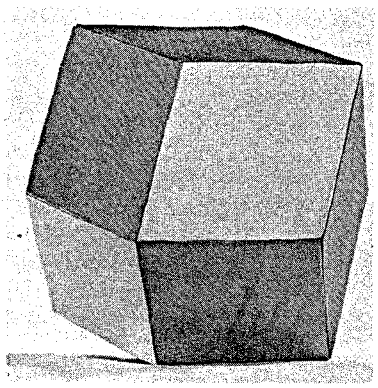
TWO INTERLOCKING
TETRAHEDRON



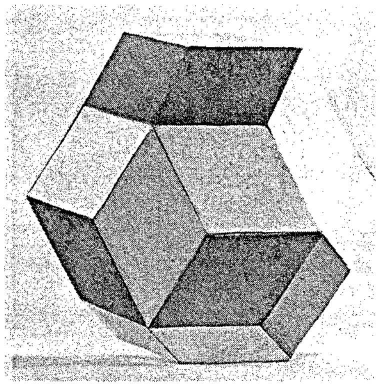
CUBE PLUS
OCTAHEDRON



DODECAHEDRON
PLUS
ICOSAHEDRON



RHOMBIC
DODECAHEDRON



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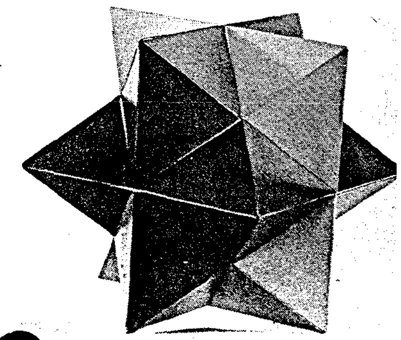
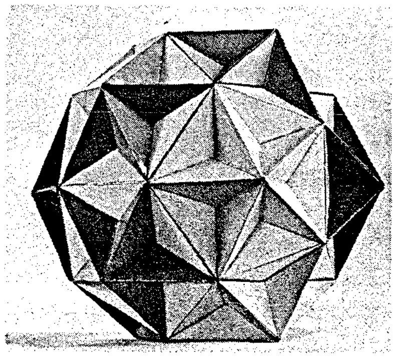


STELLATED RHOMBIC
DODECAHEDRON



STAR LINKED
TRIACONTA-
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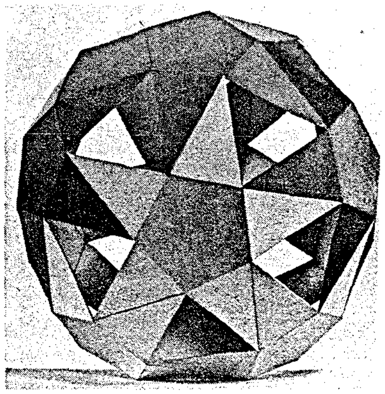
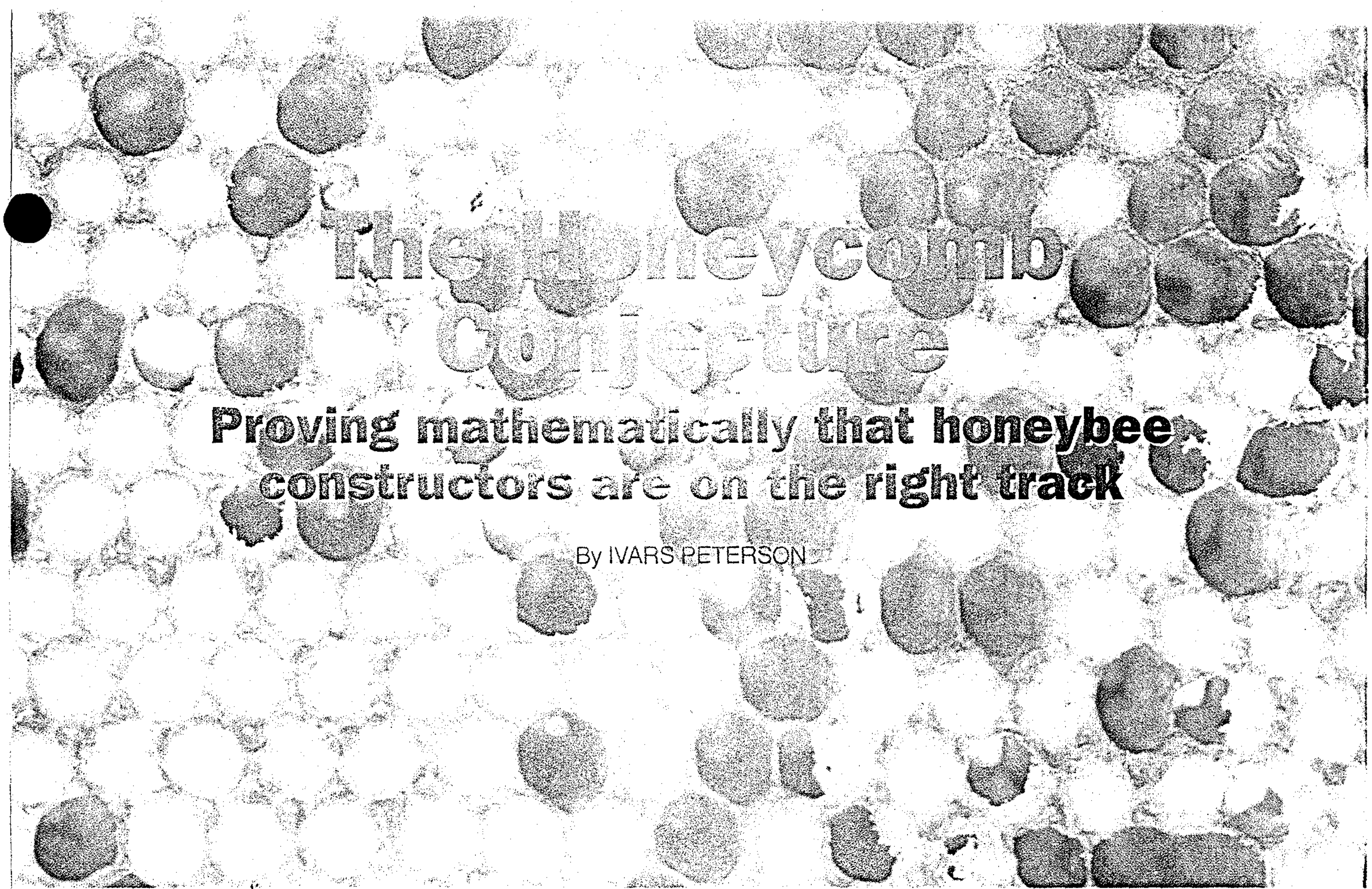


STELLAR SNUB
DODECAHEDRON

The Honeycomb Conjecture

Proving mathematically that honeybee constructors are on the right track

By IVARS PETERSON



Agricultural Research Service, USDA

The honeybee's storage system consists of an array of hexagonal cells precisely constructed from wax.

Honeybees know a thing or two about working with wax and fashioning elegant, symmetrical structures.

Gorging themselves on honey, young worker bees slowly excrete slivers of wax, each fleck about the size of a pinhead. Other workers harvest these tiny wax scales, then carefully position and mold them to assemble a vertical comb of six-sided, or hexagonal, cells. The bees cluster in large numbers, maintaining a hive temperature of 35°C, which keeps the wax firm but malleable during cell construction.

This energetic, piecemeal activity produces a strong, remarkably precise structure. Each wax partition, less than 0.1 millimeter thick, is fashioned to a tolerance of 0.002 mm. Moreover, the cell walls all stand at the correct 120° angle with respect to one another to form a lattice of regular hexagons.

Observers throughout recorded history have marveled at the hexagonal pattern of the honeybee's elaborate storage system. More than 2,000 years ago, Greek scholars commented on how bees apparently possess "a certain geometrical forethought" in achieving just the right type of enclosure to hold honey efficiently. In the 19th century, Charles Darwin described the honeycomb as a masterpiece of engineering that is "absolutely perfect in economising labour and wax."

Biologists assume that bees minimize the amount of wax they use to build their combs. But is a grid made up of regular hexagons indeed the best possible choice? What if the walls were curved rather than flat, for example?

Mathematician Thomas C. Hales of the University of Michigan at Ann Arbor has now formulated a proof of the so-called honeycomb conjecture, which holds that a hexagonal grid represents the best way to divide a surface into regions of equal area with the least total perimeter. Hales announced the feat last month and posted his proof on the Internet at http://www.math.lsa.umich.edu/~hales/.

Although widely believed and often asserted as fact, this conjecture has long eluded proof, says Frank Morgan of Williams College in Williamstown, Mass. Hales' proof "looks right to me," he comments, "although I have not checked every detail."

Last year, Hales proved Johannes Kepler's conjecture that the arrangement of the familiar piles of neatly stacked oranges at a supermarket represents the best way to pack identical spheres tightly (SN: 8/15/98, p. 103).

If Hales' proofs of the honeycomb and Kepler conjectures stand the test of time, "it's a remarkable double achievement," says physicist Denis Weaire of Trinity College Dublin in Ireland.

In an essay on the "sagacity of bees," Pappus of Alexandria noted in the fourth century A.D. how bees, possessing a divine sense of symmetry, had as their mission the fashioning of honeycombs without any cracks through which that wonderful nectar known as honey could be lost. In his mathematical analysis, he focused on the hexagonal arrangement of cells.

Although honeycomb cells are three-dimensional structures, each cell is uniform in the direction perpendicular to its base. Hence, its hexagonal cross section matters more than other factors in calculating how much wax it takes to construct a comb.

The mathematicians' honeycomb conjecture therefore concerns a two-dimensional pattern—as if bees were creating a grid for laying out tiles to cover an infinitely wide bathroom floor.

Mathematicians of ancient Greece asked what choices bees might have if they wanted to divide a flat surface into identical, equal-sided cells. Only three regular polygons pack together snugly without leaving gaps: equilateral triangles, squares, and regular hexagons.

Other polygons, such as pentagons and octagons, will not fit together without leaving spaces between the cells.

The Greeks asserted that if the same quantity of wax were used for the construction of a single three-dimensional version of the three candidate figures, the hexagonal cell would hold more honey than a triangular or square cell. Equivalently, the perimeter of a hexagonal cell enclosing a given area is less than that of a square or triangular cell enclosing the same area.

Other possibilities for arrays of cells, however, are conceivable. There's no a priori reason why the cells must all have equal sides or identical shapes and sizes. What about a crazy quilt of random polygons or cells with curved rather than straight sides?

Sorting through these alternative patterns proved a formidable task for mathematicians.

It was relatively straightforward to establish that a regular hexagon, with equal sides and 120° angles, has a smaller perimeter than any other six-sided figure of the same area. Moreover, polygons with more sides than the hexagon, such as regular octagons, do better, and polygons with fewer sides, such as squares, do worse.

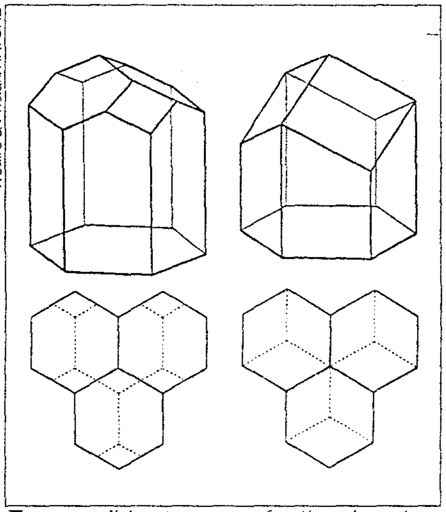
In 1943, Hungarian mathematician L. Fejes Tóth proved the honeycomb conjecture for the special case of filling the plane with any mixture of straight-sided polygons. In effect, Morgan says, Tóth established that the average number of sides per cell in a plane-filling pattern is at most six. Moreover, the advantage of having some polygons with more than six sides is less than the disadvantage of having some polygons with fewer sides. Under these conditions, the least-perimeter way to enclose and separate infinitely many regions of equal area is the regular hexagonal grid of the honeycomb.

What if cells were allowed to have curved sides? Tóth considered the question and predicted that the best answer is still a grid of regular hexagons. "Nevertheless, this conjecture has resisted all attempts at proving it," he commented.

In recent years, Morgan has refocused attention on the honeycomb conjecture and related questions, such as the most economical way of packaging a pair of identical volumes as double bubbles (SN: 8/12/95, p. 101). In the May TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, he outlined progress in proving the hexagonal honeycomb conjecture and its variants, and he suggested a possible route to a proof.

With curved sides, the complication is that a side that bulges out for one cell must bulge in for its immediate neighbor. Bulging out helps minimize the cell perimeter, while bulging in hurts.

Hales proved that the advantage of bulging out is less than the disadvantage of bulging in. "The basic idea is quite simple and elegant," says John M. Sullivan of the University of Illinois at Urbana-Champaign. Hales' main result "shows that no single cell can do better than a hexagon if appropriately penalized for having more than six sides or outward curves."



Weaire & Phelan/NATURE

Two possible structures for the closed end of a honeycomb cell. Mathematician L. Fejes Tóth showed that an end cap consisting of two hexagons and two squares (top left) requires a little less wax than the one honeybees make, with three diamond-shaped, or rhombic, panels (top right). A honeycomb consists of two layers of such cells placed back to back so that a chamber on one side is offset from its partner on the other side (bottom).

Therefore, straight-sided polygons work better than curved ones, and regular hexagons are truly best of all.

Bees have that aspect of their honeycomb structures down pat.

There's more to a honeycomb than a vertical, hexagonal grid, however. It actually consists of two layers of cells placed back to back. The cells themselves are tilted upward at an angle of about 13° from the horizontal—just enough to prevent stored honey from dripping out.

Instead of a flat bottom, each cell ends in three four-sided, diamond-shaped panels, meeting in a point like a pencil sharpened with only three knife strokes. The cells of the two layers are offset so the center of a chamber on one side is the corner of three adjacent cells on the other side. This allows the layers to interlock like the bottoms of two egg cartons fitted together. In the honeycomb, however, one layer of material serves as the bottoms of two cells. In cross section, the interface between the two layers has a zigzag structure.

The angles of each diamond-shaped, or rhombic, face of the cell bottom are 109.5° and 70.5°. In the 18th century, mathematicians proved that these particular angles give the maximum volume for a three-rhombus configuration.

In 1964, Tóth discovered that a combination of two hexagons and two squares does a little better than an end cap of three rhombuses in terms of the efficient use of wax. The difference, however, is very small. "By building such cells, the bees would save per cell less than 0.35 percent of the area of an opening (and a much smaller percentage of the surface area of a cell)," he concluded.

Several years ago, Weaire and his colleague Robert Phelan experimented with a liquid-air foam to test Tóth's mathematical model. They pumped equal-sized bubbles, about 2 mm in diameter, of a detergent solution between two glass plates to generate a double layer.

The two layers of trapped bubbles formed hexagonal patterns at the glass plates. The interface between the two layers adopted Tóth's structure.

When Weaire and Phelan thickened the bubble walls by adding more liquid, however, they unexpectedly found an abrupt transition. When the walls reached a particular thickness, the interface suddenly switched to the three-rhombus configuration of a honeycomb.

The switch also occurs in the reverse direction as liquid is removed.

So, honeybees may very well have found the optimal design solution for the thicker wax walls of their honeycomb cells.

For mathematicians, however, "many questions remain open," Morgan says.

In two dimensions, for example, mathematicians can consider what happens when they allow arrangements that include regions of several, intermingled components or empty spaces between cells. In three dimensions, the question of what space-filling arrangement of cells of equal size has the minimum surface area is still not settled (SN: 3/5/94, p. 149).

"The strategies I developed for the Kepler conjecture were very useful with the honeycomb conjecture," Hales says. "A topic for future research might be to determine to what extent [those methods] can be adapted to other optimization problems."

These are matters that concern not only mathematicians but also researchers interested in the characteristics and behavior of fluids, bubbles, foams, crystals, and a variety of biological structures, from cell assemblages to plant tissue.

"Cell and tissue, shell and bone, leaf and flower, are so many portions of matter, and it is obedience to the laws of physics that their particles have been moved, moulded and conformed," D'Arcy W. Thompson wrote in his celebrated book *On Growth and Form*, first published in 1917. "Their problems of form are in the first instance mathematical problems, their problems of growth are essentially physical problems."

The honeybee's honeycomb fits neatly into the atlas of mathematically optimal forms found in nature. □

MATHEMATICAL RECREATIONS

by Ian Stewart

The Art of Elegant Tiling

Mathematics and art have many points of contact, but none is more beautiful than the concept of symmetry. The mathematician's approach to symmetry is a little too rigid for most forms of visual art, but it can be readily applied to any art form that features repetitive patterns. Wallpaper, fabrics and tiles are familiar examples, and all of them can rise to great artistic heights. Tiles and wallpaper designed by 19th-century British artist William Morris are displayed in London's Victoria and Albert Museum. The Edo-Tokyo Museum possesses some absolutely outstanding examples of patterned kimonos, and the Alhambra palace in Granada, Spain, is renowned worldwide for its intricate tiled patterns.

Although the basic mathematics of symmetry and tilings was worked out long ago, new discoveries continue to be made, often by artists. Rosemary Grazebrook, a contemporary British artist, has invented a remarkably simple tiling system that is eminently practical and different enough from the usual rectangular tiles to be interesting. It is also ingenious and, in the right hands, beautiful.

The mathematical definition of symmetry is simple but subtle. A symmetry of a design is a transformation that leaves the design unchanged. For example, the transformation "rotate by 90 degrees" leaves a square unchanged; the transformation "reflect from left to right" leaves the human form (superficially) unchanged. A design may have many different symmetries: together they constitute its symmetry group.

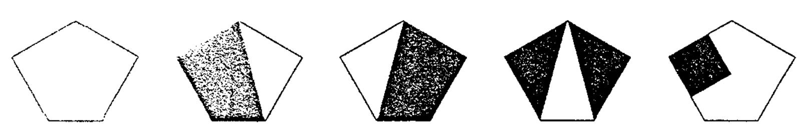
There are also many kinds of tilings. The type that has traditionally attracted the most interest from mathematicians is based on a two-dimensional lattice—in effect, a planar crystal. Ironically, the math here was first worked out in the hugely difficult case of three dimensions and only much later carried through in two dimensions. In 1891 Russian crystallographer E. S. Fedorov proved that lattices in the plane fall into 17 distinct symmetry types [*see illustration on page 98*]. The same goes for wallpaper designs and textile patterns. It may seem strange to say this when any home improvement store can show you dozens of thick books of wallpaper samples and rack after rack of tiles. In most cases, however, the differences lie in such features as color, texture and the nature of the underlying design elements. Important as these are to the customer, they do not affect the symmetry of the pattern, except that they may be constrained by it. For instance, square bathroom tiles bearing an image of a duck will have the same symmetry as similar tiles with the image of a length of seaweed—unless extra symmetry occurs in the images themselves.

Some patterns do not possess any great degree of symmetry, and these I shall ignore here. Among them are important modern discoveries such as the famous Penrose tiles, which completely cover the plane but never repeat exactly the same arrangement. The patterns of concern here are based around one "fundamental region"—a design that is repeated indefinitely in two independent directions. For example, imagine an array of standard square tiles, as seen in so many bathrooms. Our imaginary bathroom, however, has infinitely large walls, so the pattern never stops. Pick some tile. The pattern of that tile repeats in both the horizontal and vertical directions and in combinations of those. In fact, if you displace the tile by any whole number of tile widths horizontally, to the left or the right, and then by any whole number of tile widths vertically, up or down, you'll find an identical tile. So the pattern repeats in two distinct directions. Here those directions happen to be at right angles to each other, but this is not a general requirement.

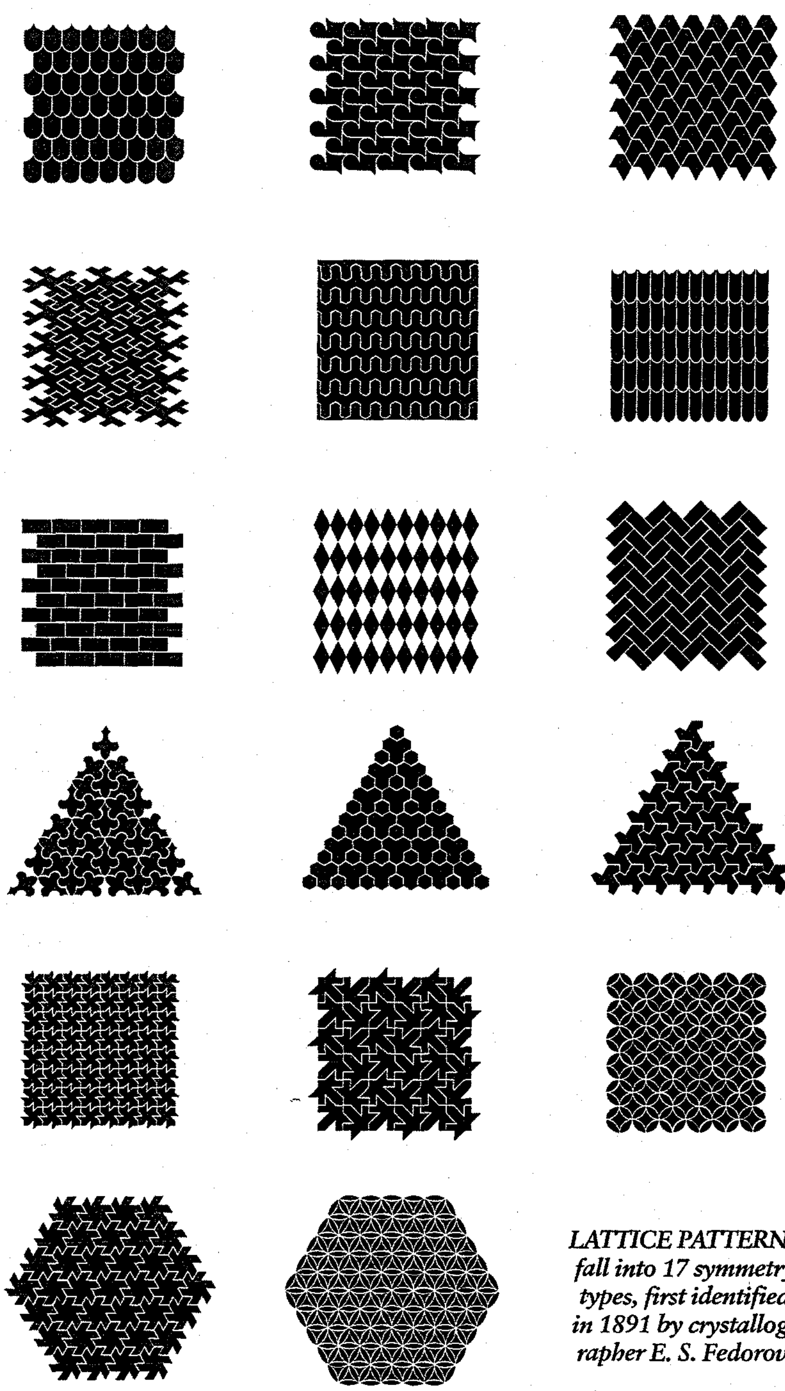
The existence of two such directions is what we mean by a lattice. Lattice symmetry is natural for wallpaper and textiles because they are usually made by forming a long roll of material along which the same pattern repeats over and over again—perhaps printed by a revolving drum or woven by a machine that repeats a fixed loop. When the paper is stuck to a wall or if the material is sewn together to cover a wider region, it is usual to match the pattern along the join. But this matching may involve what interior decorators call a "drop": you slide the paper sideways and then up or down by some amount. If there is a drop, then the lattice repeats along two directions that are not at right angles.

The lattice condition is less natural for tiles, which are made individually, but it is an easy scheme for an artist to follow when placing them on a wall or a floor. The square bathroom-tile lattice, for example, has rotational symmetries through 90 degrees. It also has reflectional symmetries about vertical, horizontal and diagonal lines that pass through the center or vertex of each tile or through the midpoint of each tile edge. A "honeycomb" tiling by regular hexagons is also a lattice, but it has different symmetries, notably rotations through 60 degrees. For a more detailed discussion of lattice patterns, see *Symmetry in Chaos,* by Michael Field and Martin Golubitsky (Oxford University Press, 1992).

Grazebrook discovered that a partic-



PENTAGONAL TILES,
colored in the patterns shown above, can form a lattice tiling in conjunction with regular hexagons (opposite page, top) *or by themselves* (opposite page, bottom).



BRYAN CHRISTIE

LATTICE PATTERNS fall into 17 symmetry types, first identified in 1891 by crystallographer E. S. Fedorov.

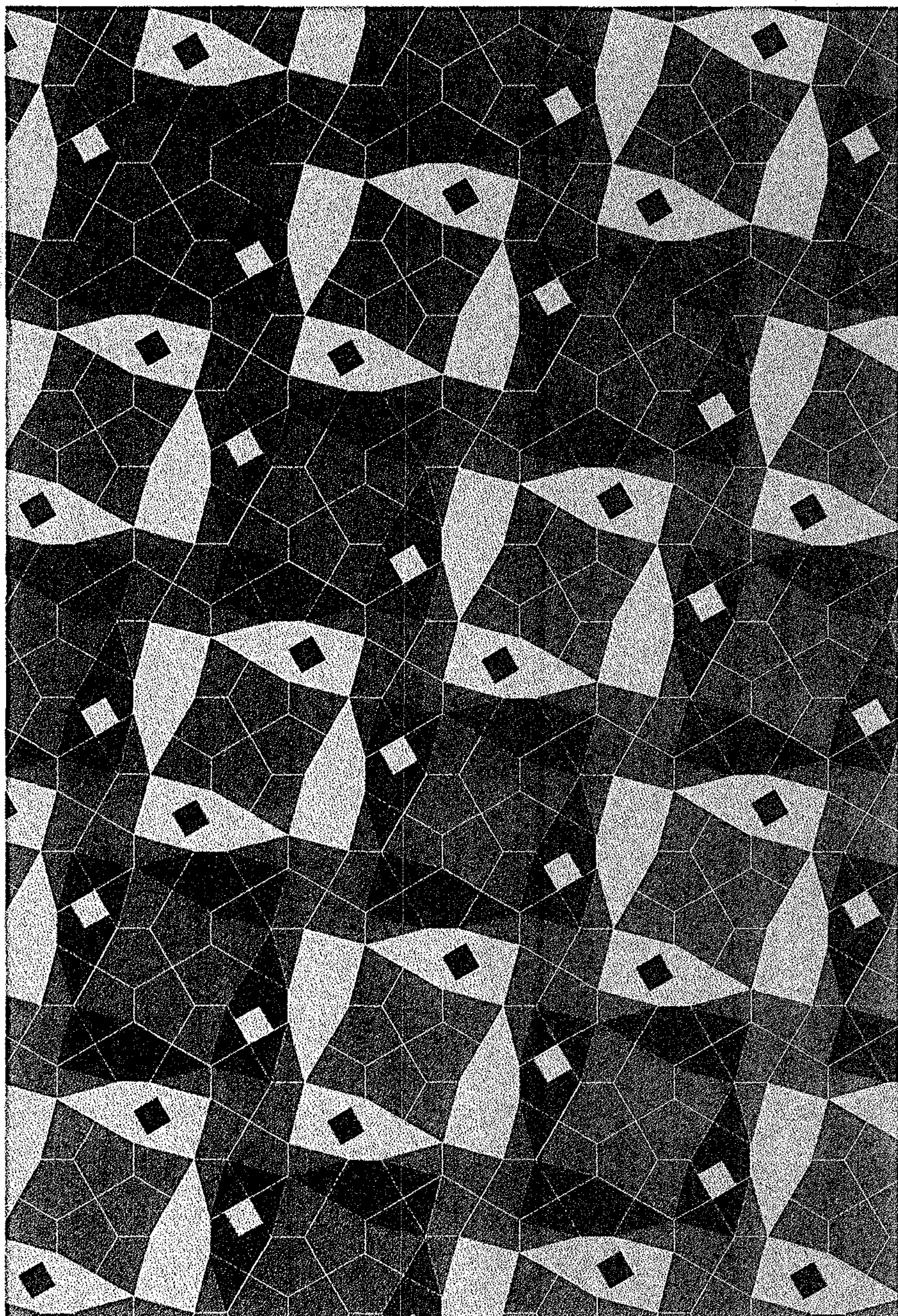
ular pentagonal tile can be the building block of a multitude of lattice patterns. A key feature of the tile is that it has two angles of 90 degrees and three of 120 degrees, allowing the tiles to be arranged in both square and hexagonal lattices [*see illustration on preceding page*]. A square tile, in contrast, has only 90-degree angles, so it can form just a few distinct lattices. Four of Grazebrook's pentagonal tiles can be fitted together to make a wide, short hexagon, which tiles the plane like bricks in a wall. When the pentagonal tiles are augmented with regular hexagons, they can form all but one of the 17 symmetry types of lattice patterns. (I leave readers the pleasure of discovering which is the missing symmetry type and how to obtain the other 16.)

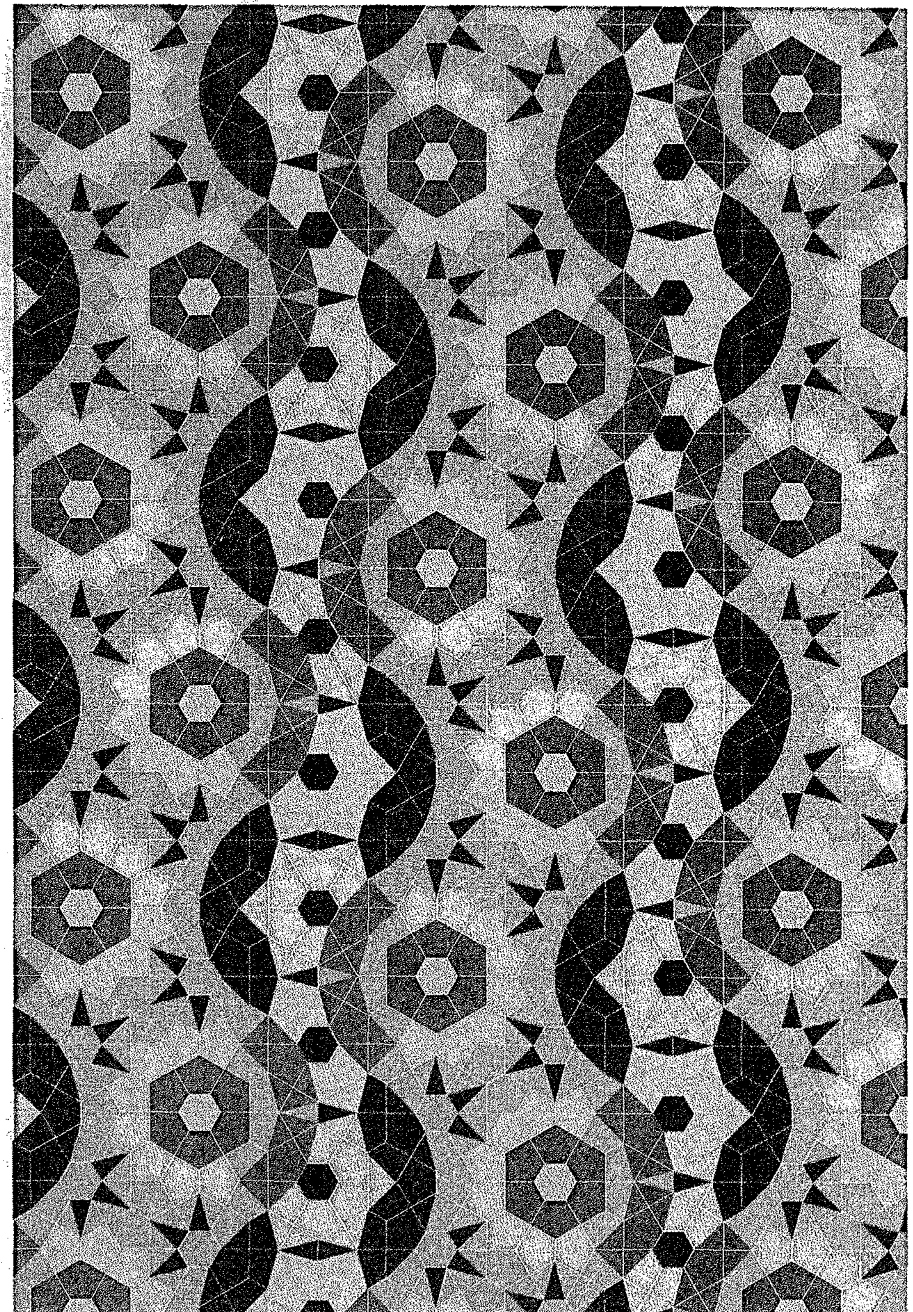
Grazebrook first got the idea for these tiles from this very column—or, more accurately, from its predecessor, Martin Gardner's inimitable Mathematical Games column. She was studying for a Ph.D. at London's Royal College of Art, focusing on the Islamic art at the Alhambra. She started a dissertation entitled "From Islam to Escher and Onwards ...". (Readers are probably familiar with the remarkable drawings of M. C. Escher, many of which use animal shapes as tiles, arranged in mathematical patterns.) Grazebrook sensed a connection between Islamic art and Escher's characteristic tiling patterns, but only after reading Gardner's column did she realize that the link is the theory of the 17 lattice symmetry types. From that point on, she began to explore ways to make Islamic patterns using various lattice-based grids.

Grazebrook introduced two distinct schemes for coloring her pentagonal tiles. One scheme divides the tile into three triangles: this is called the "Pentland" set. The other coloring scheme divides the pentagon into four regions: two squares, one kite-shaped quadrilateral and a smaller pentagon. This is the "Penthouse" set. Of course, it is possible to divide and color the tiles in many other ways, but these sets alone can form an amazing variety of designs. The designs shown on the preceding page are copyrighted, and the coloring schemes are registered. To inquire about the rights for their use, tile manufacturers can contact Grazebrook at P. O. Box 328 Isleworth, TW7 6FB, U.K. SA

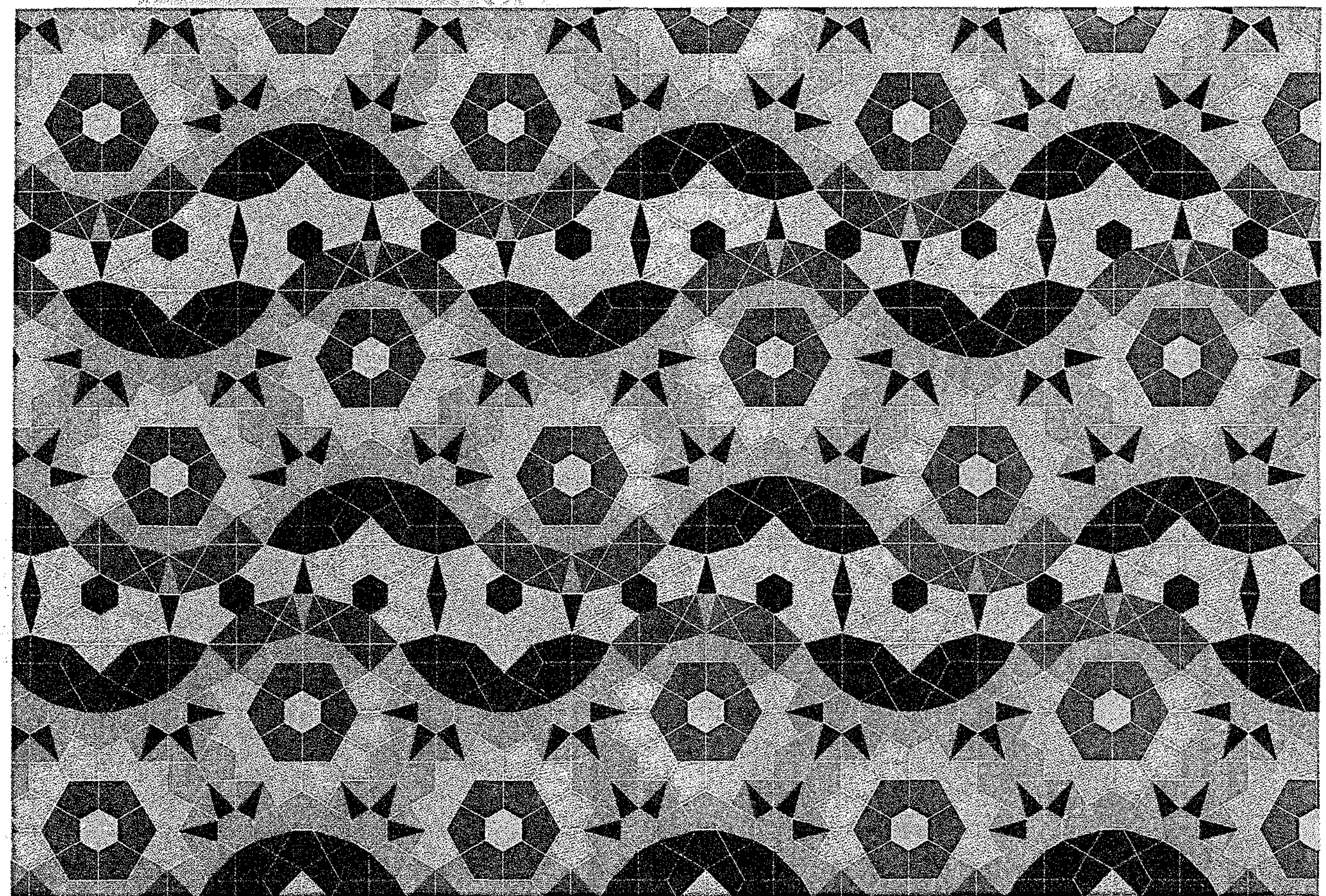
FEEDBACK

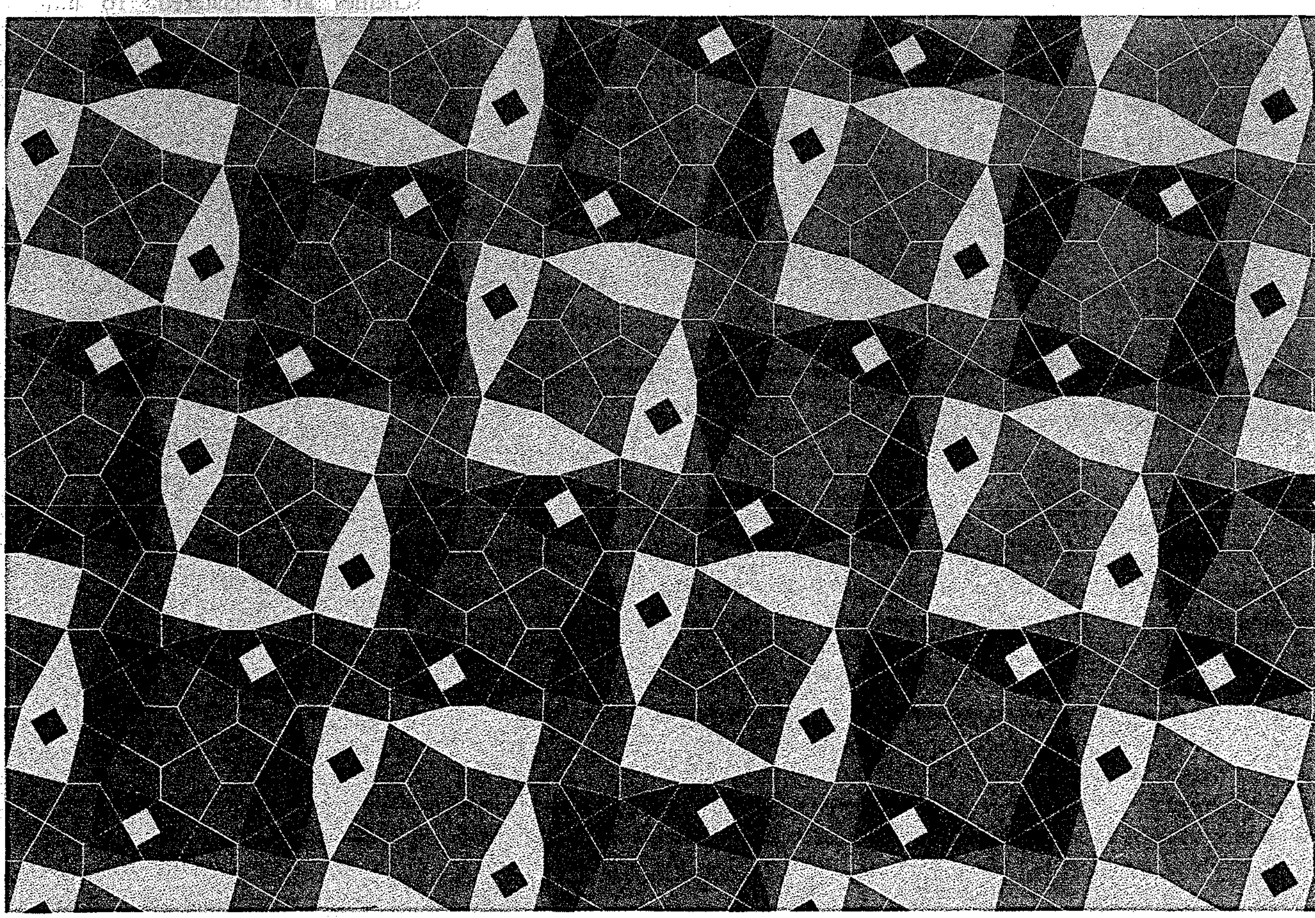
The column on coin tosses and dice ["Repealing the Law of Averages," April 1998] attracted the attention of Tom Guldbrandsen of Lyngby, Denmark. Suppose you keep rolling a die and observe the number of rolls that result in 1, 2, 3, 4, 5 or 6. What is the probability that at some stage all six totals are the same? Guldbrandsen noted that this event can happen only on rolls 6, 12, 18 and so on—multiples of 6. He found a formula for the probability that on roll 6*n* the totals are all equal. Taking account of the possibility that they may be equal more than once, he concluded that the probability is 0.021903735824 (to 12 decimal places). The analogous result for a five-sided die is 0.06469, for a four-sided die is 0.2035 and for a three-sided die is 1. —*I.S.*





BRYAN CHRISTIE; DESIGNS COURTESY OF ROSEMARY GRAZEBROOK





BRYAN CHRISTIE; DESIGNS COURTESY OF ROSEMARY GRAZEBROOK

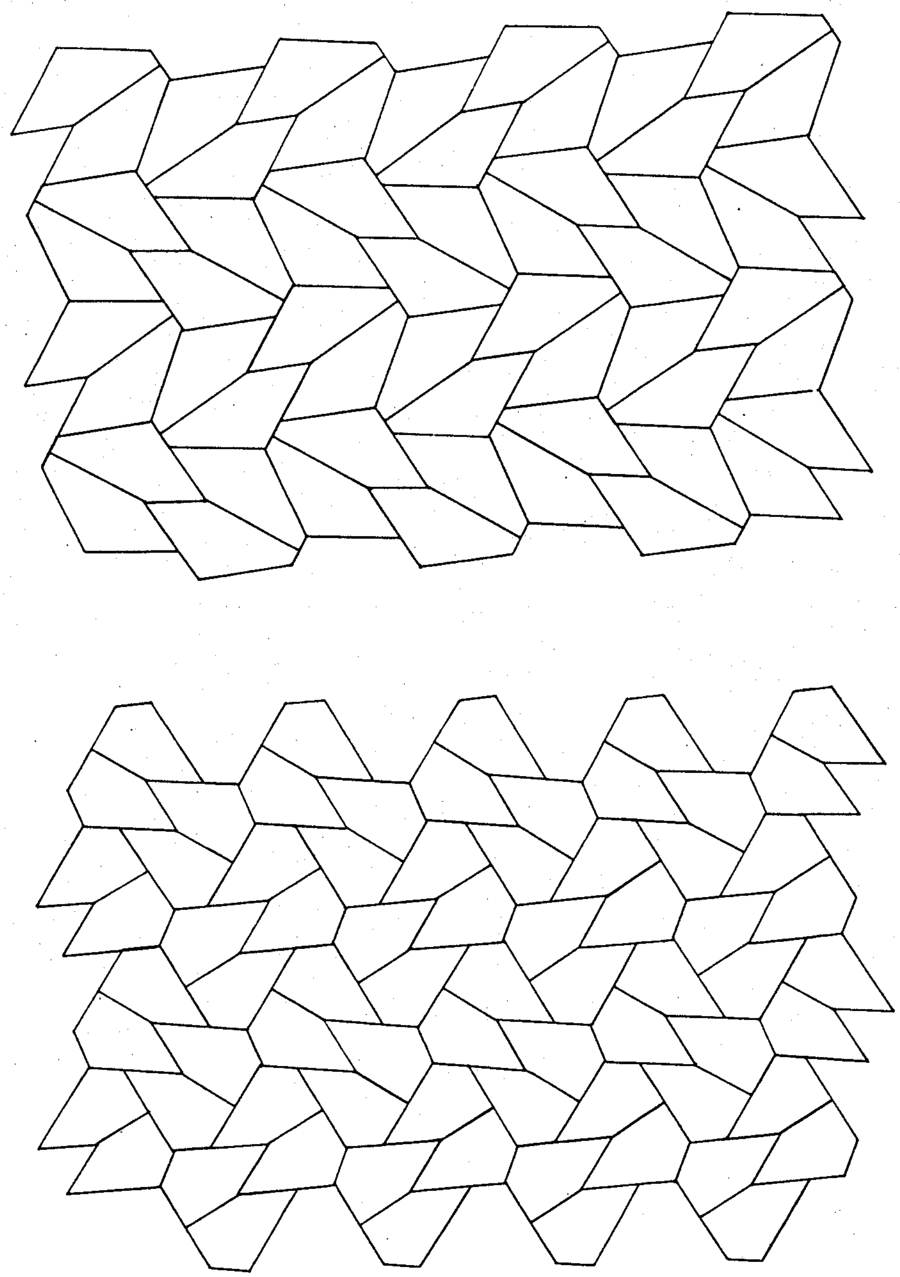


Fig. 4.6. Two examples of periodic tilings of the plane, each using a single shape (found by Majorie Rice in 1976).

Emperor's New Mind

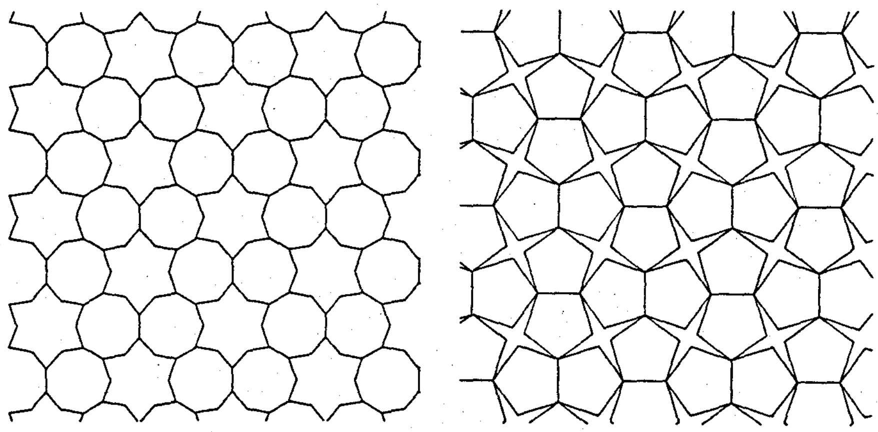


Fig. 4.7. Two examples of periodic tilings of the plane, each using two shapes.

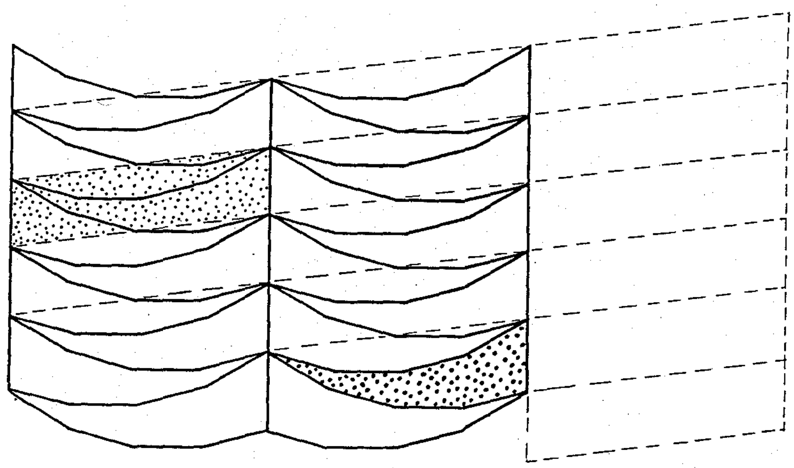


Fig. 4.8. A periodic tiling, illustrated in relation to its period parallelogram.

shared by many other single tile shapes and sets of tile shapes. Are there single tiles or sets of tiles which will tile the plane *only* non-periodically? The answer to this question is 'yes'. In Fig. 4.10, I have depicted a set of six tiles constructed by the American mathematician Raphael Robinson (1971) which will tile the entire plane, but only in a non-periodic way.

Emperor's New Mind

PENROSE TILINGS

NON PERIODIC TILINGS

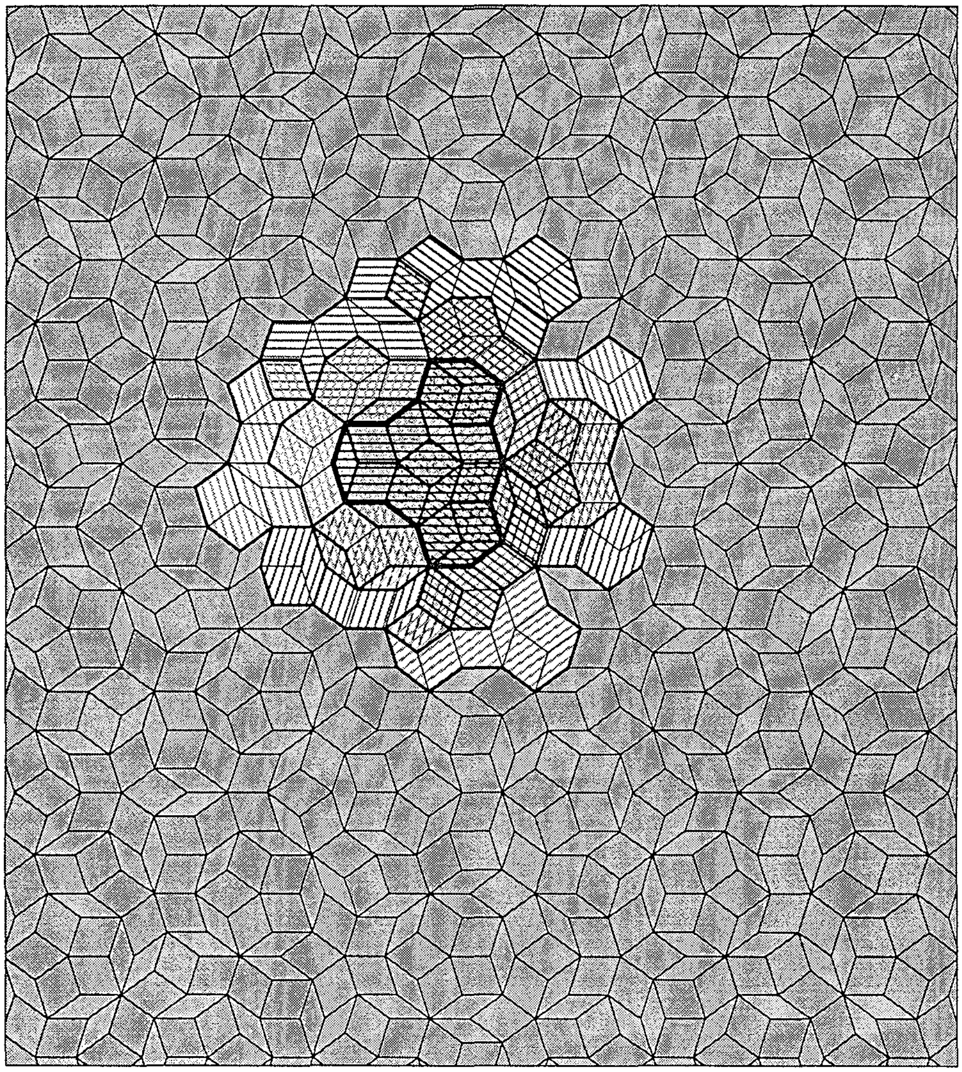
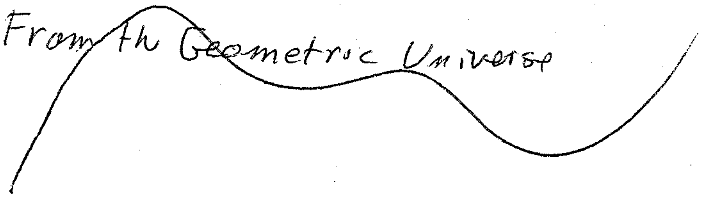


FIG. 4. A fragment of Penrose tiling in which the overlapping C-clusters are indicated with different shading in the central portion. The figure illustrates the very high C-cluster density (shown in our proof to be the maximal possible density).

From the Geometric Universe



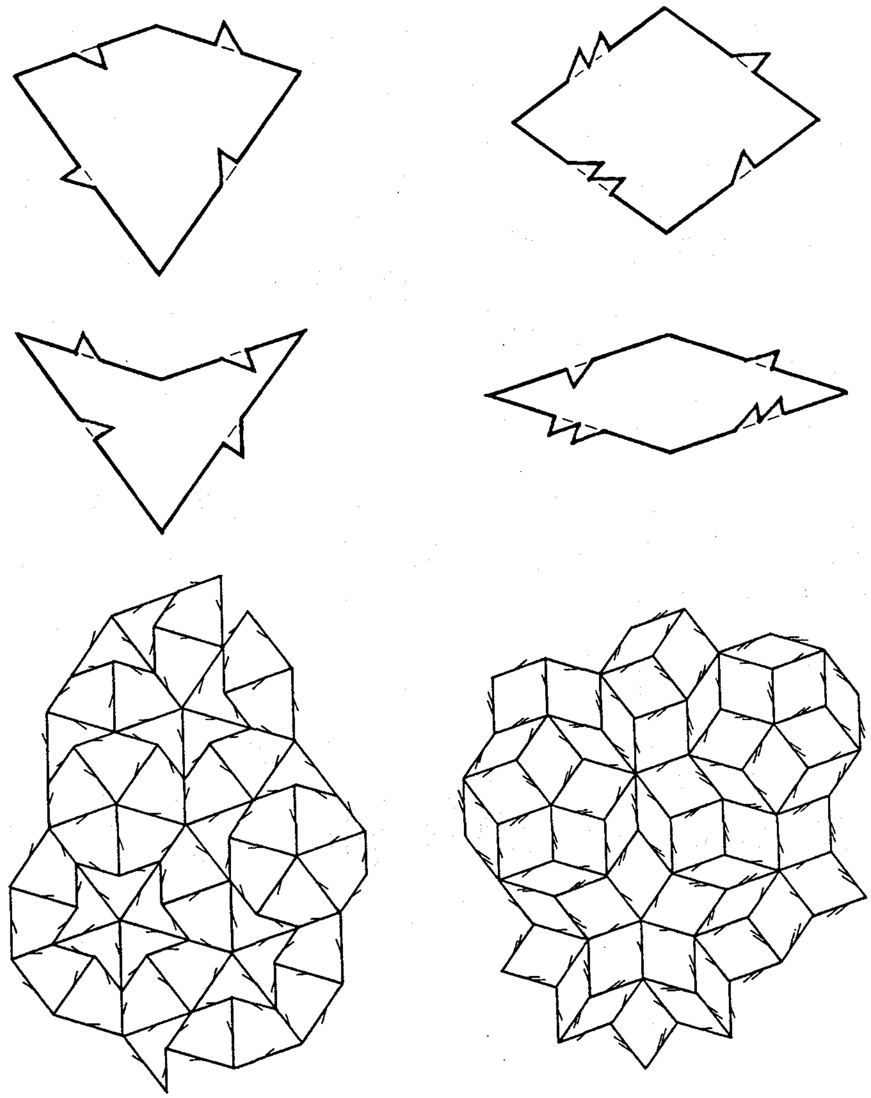


Fig. 4.12. Two pairs, each of which will tile only non-periodically ('Penrose tiles'); and regions of the plane tiled with each pair.

various operations of slicing and re-gluing, I was able to reduce it to two. Two alternative schemes are depticed in Fig. 4.12. The necessarily non-periodic patterns exhibited by the completed tilings have many remarkable properties, including a seemingly crystallographically impossible quasi-periodic structure with fivefold symmetry. I shall return to these matters later.

Emperor's New Mind

Tilings & Tessalations
Regular Polygons

I 1 species
∃ 3 tilings: triangles, squares, Hexagons
$\frac{360}{M}$ = 3, 4, 6

II Semi Tessalations
More than 1 regular polygon
Arrangements at every vertex the same
∃ 8

III Demi Tessalations More than 1 regular Polygon
2 or 3 arrangements at ~~each~~ vertexes
∃ 14

IV Irregulars V Escher VI Penrose

How are Tilings related to primes?
— to music?
Rose Windows

Islam knew Allah was mathematician

Groups

Methods of generation:

Transformations
Translation
Flip - Symmetry
Rotation
Glide [flip + rot]

The 3 single Tessalations

4.4.4.4	(16)
6.6.6	(18)
3.3.3.3.3.3	(18)

14 - DEMI's

3 ~ 60
4 ~ 90
5 ~ 108
6 ~ 120
7 ~
8 ~ 135
9 ~ 140
10 ~ 144
11 ~
12 ~ 150

5.10.5
12.4.3.3
don't work

18 ~ 160

The Eight Semi-Tessalations

Σ = 360

	8.4.8 ✓	(20)
	6.3.3.3.3	(18)
	6.3.6.3	(18)
	12.6.4 ✓	(22)
OK	6.4.3.4 ✓	(17)
	12.12.3 ✓	(25)
no	~~[illegible]~~	(20) ?
no	~~[illegible]~~	(22) ?
	4.4.3.3.3 ✓	(17)

18.9.3 ? no (30)

Subject: RE: Birthday
Date: Tue, 6 Oct 1998 10:37:57 -0400
X-Mailer: Internet Mail Service (5.5.2232.9)

References: Web site
Bob Williams
Polytopes

Rules for limiting possibilities.

1. 1 species of polygon
   Result 3 tilings "regular"
2. 1 species of vertex, but any variety of polygons
   Result 8 semi-regular tilings
3. 2 or 3 species of vertex
   Result 14 demi-regular tilings

systems adding to 360° but which do not tile

5.10.5 — 24.8.3

12.4.3.3 no — 12.4.6 OK

18.9.3 — 15.10.3

20.5.4

n	$180\frac{n-2}{n}$
3	60
4	90
5	108
6	120
8	135
9	140
10	144
12	150
15	156
16	
18	160
20	162
24	165
30	168
36	170

13 + 12 = 26, 13

156 + 144 = 300

4.4.4.

28 × 6 = 168

24.8.3	35	21,16,5	108
20.5.4	29	16,15,1	168
18.9.3	30	15,9,6	276
15.10.3	28	12,7,5	144
12.6.4 OK	22	8,6,2	420 all even
10.5.5	20	5,5,10	
~~18.4.3~~			
3.3.3.4.4 ok	17		

168 × 2 = 336

12,12,3 — 9,9,0

384 — 440

$nx = 180m - 360$

$360 = m(180 - x)$

$m^{-1} = \frac{180-x}{360} = \frac{1}{2} - \frac{x}{360}$

$n = \frac{360}{180-x}$

3 5/3

4 8/...

2, 3

3/2, 10/3

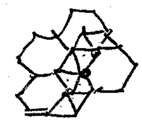
2, 3/2

2, 1	1/2
2, 3 ~~8~~	3/2
2, 2	1
3/2, 5	10/3
2, 6	3
4, 5	5/4
3, 8	8/3

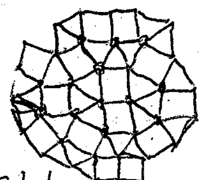
LASERJET III WP ~ 6.0

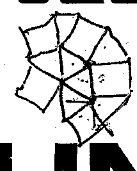
OUTLINE FONT TEST — CG TIMES 30 PT

OUTLINE FONT TEST — UNIVERS 30 PT



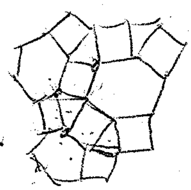
OUTLINE FONT TEST — NEW COURIER BOLD 30 PT

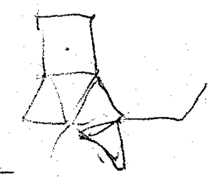




OUTLINE FONT TEST — SWISS 721 BLACK 30 PT

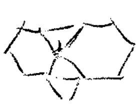
OUTLINE FONT TEST — TIMES NEW ROMAN BOLD 30 PT

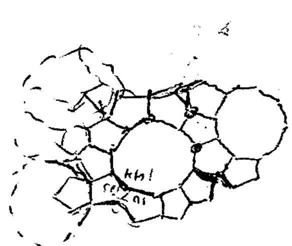


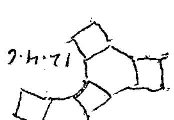


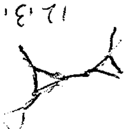
4.6.4.3

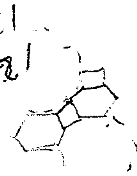
Apparently only indigonous fonts in the HP III can be → OUTLINED

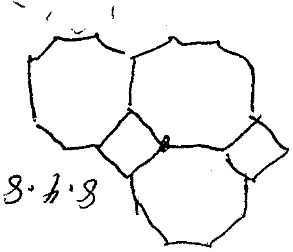




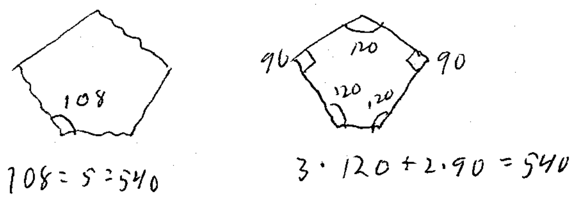


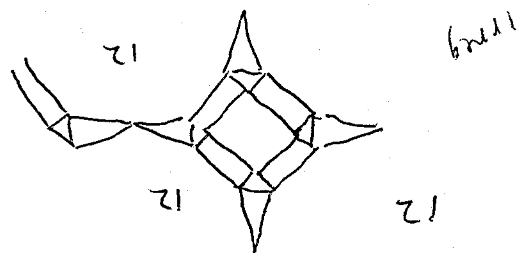


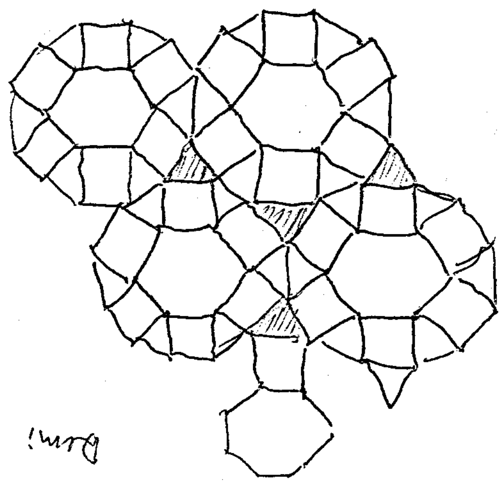


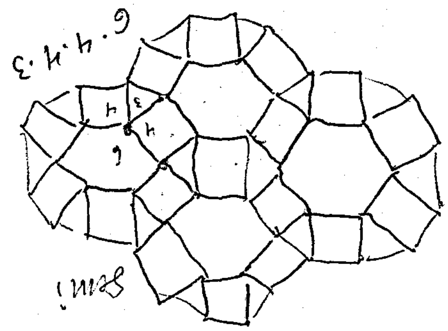


MODIFIED PENTAGON

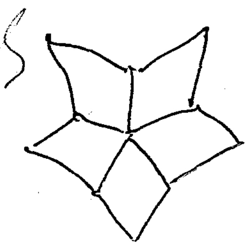


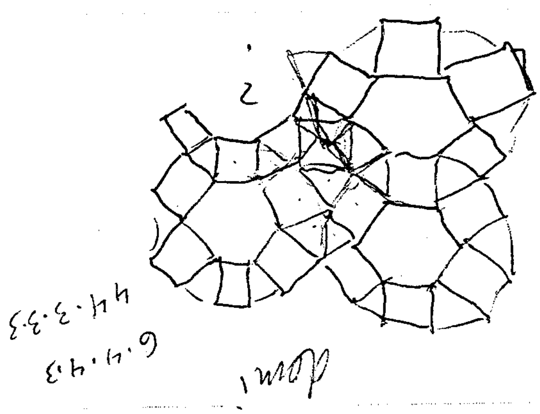


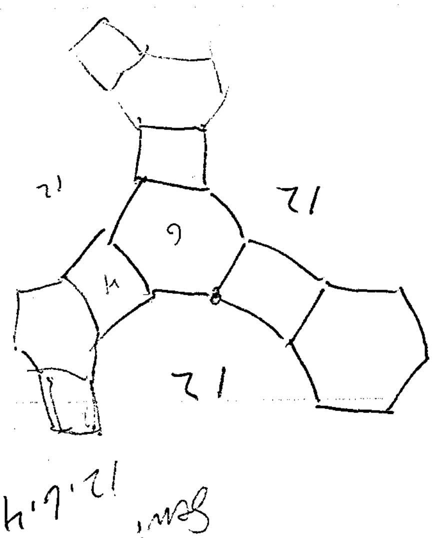




2 types of rhombus







Merry Christmas

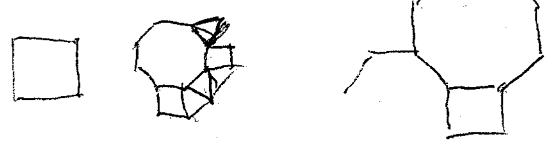
Merry Christmas

360
150
210

12
6
4
210
120
-90

12
3
3
4

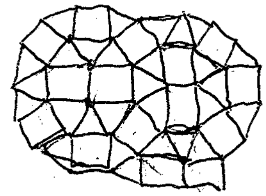
150
90



120
180
3

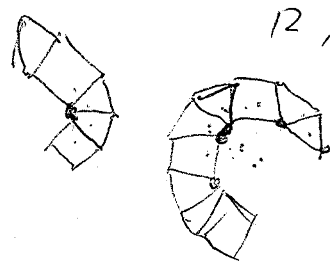
6 4 4 3

4 - 4 · 3 · 3 · 3

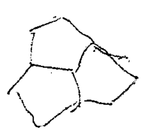


6 · 3 · 3 · 3 · 3	18
4 · 4 · 3 · 3 · 3	17
6 · 3 · 6 · 3	18
6 · 4 · 3 · 4	17
12 · 3 · 12	27
12 · 6 · 4	22
8 · 4 · 8	20

12 , 4 , 3 , 3



180
30
360
16
18
150



20

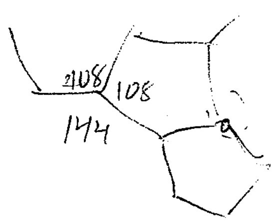
160
140
300
60

160
90
250

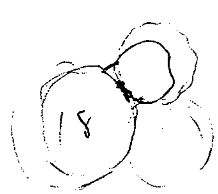
280
90
370

360
150
210
90
120

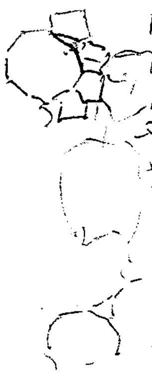
12
4
3
3



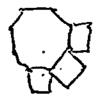
216
144
60



20
30
22



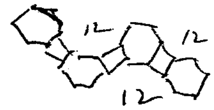
108
108
144
360



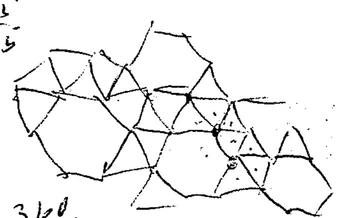
135
90
225

360
225
135

12, 4, 6
12, 4, 3 3



180
36
144



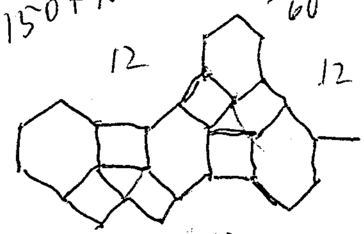
✓ Semi
6 · 3 · 3 · 3 · 3

150 + n 60 = 360
150
210
60

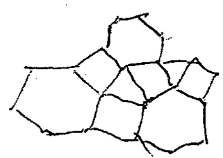
360
240
120

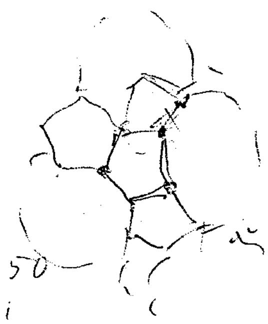
150 + 90

144
216
360



demi





150

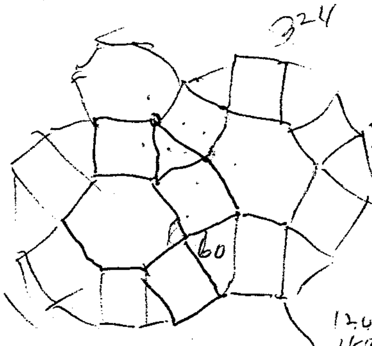
108
3
324

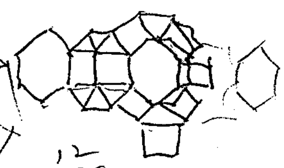
140
180
320

140
150
290

8 4

135
135
270





12

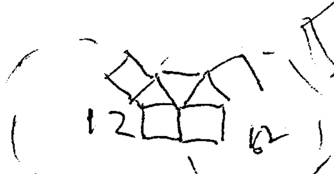
20
18 | 360

150
90
120
360

32

120
180
300

160
60
220

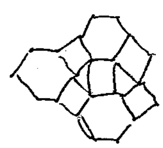


perf 12 and Hatem 12's

360
72

180
72

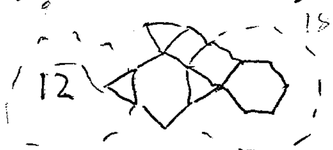
72 180
36
144



360
140
220
180

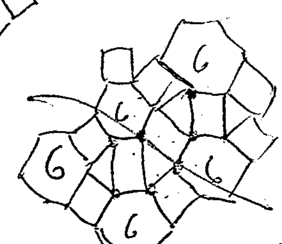
12 - 150
8 - 135
6 - 120
4 - 90
3 - 60

270



135 / 45

✓



360

3

6 · 4 · 3 · 4
Semi

12 · 6 · 4

9   140 / 40

140
60 = 200

9

360
280
80

360
140
220
90
130

160
140

$n := 3, 4 .. 20$

$F(n) := n \cdot \cos\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{\pi}{n}\right)$

F(N) = A(n)/R(n)^2

SIDE 21

n =	F(n) =	F(n + 100) =	F(n + 1000) =
3	1.2990381057	3.13964459	3.141572106
4	2	3.139681866	3.141572147
5	2.3776412907	3.139718082	3.141572188
6	2.5980762114	3.139753278	3.141572229
7	2.7364101886	3.139787493	3.141572269
8	2.8284271247	3.139820761	3.14157231
9	2.8925442436	3.139853118	3.14157235
10	2.9389262615	3.139884597	3.14157239
11	2.973524496	3.13991523	3.14157243
12	3	3.139945045	3.14157247
13	3.0207006183	3.139974073	3.14157251
14	3.0371861738	3.14000234	3.14157255
15	3.0505248231	3.140029874	3.141572589
16	3.0614674589	3.140056698	3.141572629
17	3.0705541626	3.140082838	3.141572668
18	3.0781812899	3.140108316	3.141572707
19	3.0846449574	3.140133154	3.141572746
20	3.0901699437	3.140157375	3.141572785

↓ π

$$\lim_{n \to \infty} \frac{A_n}{R_n^2} = \pi$$

i.e. for a circle $A = \pi R^2$

n = # of sides of polygon

For a square Area = $2\ R^2$

For a dodecagon Area = $3\ R^2$

POLAREA.MCD

AREAS OF REGULAR POLYGONS

SIDE = 1

n := 3, 4.. 20

$$A(n) := \frac{n}{4 \cdot \tan\left(\frac{\pi}{n}\right)}$$

$A(1) = \infty$
$A(2) = 0$

n =	A(n) =		A(n + 16) =	n + 16 =
3	0.433012702	$\sqrt{3}/4$	28.465189428	19
4	1		31.568757573	20
5	1.720477401		34.831474124	21
6	2.598076211	$3\sqrt{3}/2$	38.253340245	22
7	3.633912444		41.834356853	23
8	4.828427125	$2+\sqrt{8}$	45.574524676	24
9	6.181824194		49.473844302	25
10	7.694208843		53.532316204	26
11	9.365639907		57.749940772	27
12	11.196152423	$3\sqrt{3}+6$	62.126718325	28
13	13.185768328		66.662649129	29
14	15.334501936		71.357733407	30
15	17.642362911		76.211971345	31
16	20.109357969		81.225363101	32
17	22.735491898		86.39790881	33
18	25.520768188		91.729608587	34
19	28.465189428		97.22046253	35
20	31.568757573		102.870470725	36

(6) = 6(3)

(12) = (6) + 6(4) + 6(3)
$= \frac{3\sqrt{3}}{2} + 6 + \frac{3\sqrt{3}}{2}$
$= 6 + 3\sqrt{3}$

α = External angle of a polygon = $\frac{360}{n}$

internal angle, $\beta = 180 - \alpha$

$\beta = \frac{n-2}{n} \cdot 180$

TWO METHODS OF GROWTH

I] (12) = (6) + 6(4) + 6(3) doubles the number of side but the length of side s=1, remains constant

II] side = s
(6) + 18(3) = (6) but this hexagon has side = 2s
$\frac{3\sqrt{3}}{2} + \frac{18\sqrt{3}}{4} = 6\sqrt{3}$
$\therefore \div 4$
Number of sides constant but length of side doubled

I double number of side complexity ↑
II double length of side size ↑

SIDE = 1

P = apothem

center to side ⊥

G(n) = A(n)/P(n)^2 where P(n) = 1/4 tan ^2

$n := 3, 4 .. 20$

$G(n) := n \cdot \tan\left(\frac{\pi}{n}\right)$

n =	G(n) =	G(n + 100) =	G(n + 1000) =
3	5.196152423	3.142567229	3.141602927
4	4	3.142548571	3.141602907
5	3.63271264	3.142530443	3.141602886
6	3.464101615	3.142512826	3.141602866
7	3.371022332	3.142495701	3.141602846
8	3.313708499	3.14247905	3.141602826
9	3.275732108	3.142462855	3.141602805
10	3.249196962	3.1424471	3.141602785
11	3.229891422	3.142431769	3.141602765
12	3.215390309	3.142416847	3.141602745
13	3.204212219	3.142402319	3.141602725
14	3.195408641	3.142388173	3.141602706
15	3.188348425	3.142374394	3.141602686
16	3.182597878	3.142360969	3.141602666
17	3.177850751	3.142347888	3.141602646
18	3.173885653	3.142335138	3.141602627
19	3.170539238	3.142322709	3.141602607
20	3.167688806	3.142310588	3.141602588

↓ π

For a square
$A = 4 P^2$

lim $P \to R$
or $R \to P$

	AREA	SIDE=1 RADIUS	
(3)	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	A = R
(4)	1	$\frac{\sqrt{2}}{2}$	A = 1
(6)	$\frac{3\sqrt{3}}{2}$	1	R = 1

These are the three single polygon tilings

Necessary for tiling that

A = 1 or R = 1, or A = R ?

POLRAD.MCD

RADII OF REGULAR POLYGONS

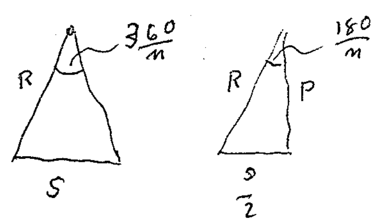
$n := 3, 4 .. 20$

$$R(n) := \frac{1}{2 \cdot \sin\left(\frac{\pi}{n}\right)}$$

SIDE = 1

R center to a vertex

n =	R(n) =	R(n + 16) =	n + 16 =
3	0.577350269	3.03776691	19
4	0.707106781	3.196226611	20
5	0.850650808	3.35475307	21
6	1	3.513337092	22
7	1.152382435	3.671971098	23
8	1.306562965	3.830648788	24
9	1.4619022	3.989364878	25
10	1.618033989	4.148114905	26
11	1.774732766	4.306895074	27
12	1.931851653	4.465702135	28
13	2.089290734	4.62453329	29
14	2.246979604	4.783386117	30
15	2.404867172	4.942258507	31
16	2.562915448	5.101148619	32
17	2.721095576	5.260054833	33
18	2.879385242	5.418975724	34
19	3.03776691	5.577910028	35
20	3.196226611	5.736856623	36



$s = 1$

$$\sin\left(\frac{180}{n}\right) = \frac{s}{2R}$$

$$\tan\left(\frac{180}{n}\right) = \frac{s}{2p} \qquad p = \frac{1}{2\tan\left(\frac{\pi}{n}\right)}$$

$$A(n) = \frac{sp}{2} n = \frac{n}{4} \frac{1}{\tan\left(\frac{\pi}{n}\right)}$$

H-SPACE

GROWTH vs MORPH — Geometric METAPHOR

$\frac{3\sqrt{3}}{2} = (6) \rightarrow ((6)) = 6\sqrt{3}$ $\qquad (6) \rightarrow (12)$

Δ only ~ the party-line

$\frac{3\sqrt{3}}{2} + \frac{18\sqrt{3}}{4} = 6\sqrt{3}$ $\qquad \frac{3\sqrt{3}}{2} + \frac{6\sqrt{3}}{4} + 6 = 3\sqrt{3} + 6 = (12)$

Area $((6)) = 6\sqrt{3} = 10.392305$ $\qquad$ Area $(12) = 11.196152$

$\frac{((6))}{(6)} = \frac{6\sqrt{3}}{\frac{3\sqrt{3}}{2}} = 4$ $\qquad \frac{(12)}{(6)} = \frac{6 + 3\sqrt{3}}{\frac{3\sqrt{3}}{2}} = 2 + \frac{4}{\sqrt{3}} = 4.309401$

$$\frac{((6))}{(12)} = \frac{6}{3 + 2\sqrt{3}}$$

$$2(12) - ((6)) = 6\sqrt{3} + 12 - 6\sqrt{3} = 12$$

$$\frac{((6))}{2} + 6 = (12)$$

$$\frac{((6))}{2} - 6 = (12) - 12$$

$$((6)) = 2(12) - 12$$

can go to (((6))) $\qquad$ (12) → (24) but not equal sides

Area Δ or $(3) = \frac{\sqrt{3}}{4}$ $\qquad 24(3) = ((6))$

$A(6) = \frac{3\sqrt{3}}{2}$ $\qquad 6(3) = (6)$

$A(4) = 1$ $\qquad 12(3) + 6(4) = (12)$

G--c table GCTABLE.MCD 07/08/01

G-c table #2

n horizontal , m vertical

$a := 10.476821$ $n := 0, 1 .. 15$

c vs G/2

$b := -3.587651$ $m := 0, 1 .. 8$

$K_{m,n} := n \cdot a - m \cdot b$

c 7

K =

		7	8	9	10	11	12
	0	73.337747	83.814568	94.291389	104.76821	115.245031	125.721852
G-½	1	76.925398	87.402219	97.87904	108.355861	118.832682	129.309503
G-1	2	80.513049	90.98987	101.466691	111.943512	122.420333	132.897154
	3	84.1007	94.577521	105.054342	115.531163	126.007984	136.484805
	4	87.688351	98.165172	108.641993	119.118814	129.595635	140.072456
	5	91.276002	101.752823	112.229644	122.706465	133.183286	143.660107
	6	94.863653	105.340474	115.817295	126.294116	136.770937	147.247758
	7	98.451304	108.928125	119.404946	129.881767	140.358588	150.835409
	8	102.038955	112.515776	122.992597	133.469418	143.946239	154.42306