FULCRUM NUMBERS

NOTATIONS

ORIGINAL

B = FULCRUM NUMBER

L = number of numbers in series < B

U = number of numbers in series > B

D = smallest number in L series

H = largest number in U series

$E = L - U$

OTHER NOTATIONS [WHY?]

$E \rightarrow P$

$U \rightarrow K$

T = ?

J = L

G = ?

$S = Z q U = Z q L$

Q = ?

P =? $\frac{L}{U}$

FULCRUM NUMBERS

Number Theory is the branch of mathematics having to do with the properties of the positive integers, often referred to as *natural* numbers. Number theory is devoted to the relations between integers and sub-classes of integers, such as *perfect* numbers, *prime* numbers, *Pythagorean* numbers, *Fibonacci* numbers, etc, etc. In this essay the properties of a special class of integers called *fulcrum* numbers will be investigated. Fulcrum numbers are those integers, **B**, which "balance" the sum of a sequence of integers immediately less than **B** with the sum of a sequence of integers immediately greater than **B.**

Examples: The number 16 is a fulcrum number since the sum of the four integers less than 16 is equal to the sum of the three integers greater than 16.

$$54 = 12 + 13 + 14 + 15 \quad [\mathbf{B = 16}] \quad 17 + 18 + 19 = 54 \qquad \text{also,}$$

$$100 = 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 \quad [\mathbf{B = 17}] \quad 18 + 19 + 20 + 21 + 22 = 100$$

In general, an integer is a fulcrum number, **B**, when,

$$\mathbf{1)} \qquad \sum_{\mathbf{D}}^{\mathbf{B-1}} \mathbf{n} = \sum_{\mathbf{B+1}}^{\mathbf{H}} \mathbf{n}$$

where D is the least integer in the lower series and H is the greatest integer in the upper series. Setting D = B – L and H = B + U, where L is the number of numbers in the lower series and U is the number of numbers in the upper series, equation 1) becomes,

$$\mathbf{2)} \qquad \sum_{\mathbf{B-L}}^{\mathbf{B-1}} \mathbf{n} = \sum_{\mathbf{B+1}}^{\mathbf{B+U}} \mathbf{n}$$

This equation may be rewritten,

$$\mathbf{3)} \qquad \sum_{1}^{\mathbf{B-1}} \mathbf{n} - \sum_{1}^{\mathbf{B-1-L}} \mathbf{n} = \sum_{1}^{\mathbf{B+U}} \mathbf{n} - \sum_{1}^{\mathbf{B}} \mathbf{n}$$

Using the general summation formula: $\sum_{n=1}^{N} \mathbf{n} = \mathbf{N(N+1)/2}$

Equation **3)** becomes:

$$\mathbf{4)} \qquad \mathbf{2B(L\text{-}U) = U^2 + U + L^2 + L}$$

Setting **E = (L-U)**, in equation 4) gives:

$$5) \qquad \mathbf{B - U = \frac{U(U+1)}{E} + \frac{E+1}{2}}$$

as the formula for fulcrum numbers in terms of U and E.

CASE I. E = 1
When E = 1, equation 5) becomes,

$$\mathbf{B = U^2 + 2U + 1 = (U + 1)^2}$$

Which says that B is a perfect square for all values of U. That is, all integers that are perfect squares, 4, 9, 16, 25, 36.... are fulcrum numbers.

CASE II. E = 2
When the lower sequence is **two** greater than the upper sequence, equation **5)** becomes

$$\mathbf{B - U = \frac{U(U + 1)}{2} + \frac{3}{2}}$$

Whatever integer value of U, the right member can never be an integer. U(U+1) will be even whether U is even or odd. Hence U(U+1)/2 will be an integer. This leaves the non-integer value 3/2 in the right member. Therefore there can be no fulcrum numbers when E = 2.

CASE III. E = an odd integer. 3, 5, 7, 9, 11, 13, ...
Equation 5) takes the form,

$$\mathbf{B - U = \frac{EVEN}{ODD} + \frac{EVEN}{2}}$$

when the first term on the right is divisible it will be an even integer. The second term is always divisible and therefore an integer. Hence, whenever E is an odd integer, fulcrum numbers are possible.

CASE IV. E = an "even-even" integer, that is an even integer which divided by 2 is still even.
E = 4, 8, 12, 16, 20, 24, ... = 4• N, where N is any integer.

$$\mathbf{B - U = \frac{EVEN}{4 \bullet N} + \frac{ODD}{2} = \frac{EVEN + 2 \bullet N \bullet ODD}{4 \bullet N} = \frac{EVEN}{EVEN}}$$

The numerator on the right is even, hence fulcrum numbers are possible whenever E = 4•N.

CASE V. E = an "odd-even" integer, that is an even integer which when divided by 2 is odd.
E = 2, 6, 10, 14, 18, 22, ... = 2 + 4•N, where N is any integer.

$$\mathbf{B - U = \frac{EVEN}{2+4 \bullet N} + \frac{ODD}{2} = \frac{EVEN + (1+2 \bullet N) \bullet ODD}{2+4 \bullet N} = \frac{EVEN + ODD + EVEN}{EVEN} = \frac{ODD}{EVEN}}$$

Since ODD/EVEN can never be an integer, there are no fulcrum numbers for E = 2 + 4•N.

ODD VALUES OF E:
To derive specific formulae for values of E $B = B(E,N)$

Next, return to equation 5) and rewrite it in the form,

$$B \bullet E = U^2 + U(E+1) + E(E+1)/2$$

If we set $U = E$, then $B = (5E+3)/2$ and if we set $U = E - 1$, then $B = (5E-3)/2$
In both cases whenever E is odd, B will be an integer.

If we set $U = 2E$, then $B = (13E+5)/2$ and if we set $U = 2E - 1$, then $B = (13E-5)/2$
Again, In both cases whenever E is odd, B will be an integer.

Similarly, Setting: $U = 3E$ and $3E-1$ gives $B = (25E \pm 7)/2$
$U = 4E$ and $4E-1$ gives $B = (41E \pm 9)/2$
$U = 5E$ and $5E-1$ gives $B = (61E \pm 11)/2$

In general for $U = N \bullet E$ and $N \bullet E - 1$,

6)
$$B = \frac{(2N^2 + 2N + 1)E \pm (2N + 1)}{2}$$

It is evident from equation **6)** that for E <u>odd</u> and for any integer value of N, B will be an integer. This function $B = B(E,N)$ gives all the fulcrum numbers possible for odd values of E, and any positive integers, N.

Specific examples:

	+ case	- case	98
$E = 1$	$B = N^2 + 2N + 1$ [4,9,16,25...]	$B = N^2$ [1,4,9,16...]	$\Delta_2 = 2$
$E = 3$	$B = 3N^2 + 4N + 2$ [9,22,41,66...]	$B = 3N^2 + 2N + 1$ [6,17,34,57...]	$\Delta_2 = 6$
$E = 5$	$B = 5N^2 + 6N + 3$ [14,35,66,107]	$B = 5N^2 + 4N + 2$ [11,30,59,95..]	$\Delta_2 = 10$
$E = 7$	$B = 7N^2 + 8N + 4$ [19,48,91,148]	$B = 7N^2 + 6N + 3$ [16,43,84,139]	$\Delta_2 = 14$

EVEN VALUES OF E:
It was shown above that for fulcrum numbers to exist when E is even, that E must equal 4N.
The counter function of **6)** for $E = 4N$ is,
corresponding

7)
$$B = E\,N^2 \pm N + E/4$$

Specific examples:	+ case	- case	
$E = 4$	$B = 4N^2 + N + 1$ [6,19,40,69]	$B = 4N^2 - N + 1$ [4,15,34,61]	$\Delta_2 = 8$
$E = 8$	$B = 8N^2 + N + 2$ [11,36,77,134]	$B = 8N^2 - N + 2$ [9,32,71,126]	$\Delta_2 = 16$
$E = 12$	$B = 12N^2 + N + 3$ [16,53,114,199]	$B = 12N^2 - N + 3$ [14,49,108,191]	$\Delta_2 = 24$
$E = 16$	$B = 16N^2 + N + 4$[21,70,151,264]	$B = 16N^2 - N + 4$ [19,66,145,256]	$\Delta_2 = 32$

NOTATION

B = fulcrum Number

L = number of numbers in the lower series

U = number of numbers in the upper series

D = smallest number in lower series

H = largest number in upper series

E = L - U

DIOPHANTINE ARITHMETIC

ADDITION	EVEN + EVEN = EVEN	EVEN + ODD = ODD	ODD + ODD = EVEN
SUBTRACTION	EVEN - EVEN = EVEN	EVEN - ODD = ODD	ODD - ODD = EVEN
MULTIPLICATION	EVEN x EVEN = EVEN	EVEN x ODD = EVEN	ODD x ODD = ODD
DIVISION	EVEN / EVEN = EITHER	EVEN / ODD = EVEN	ODD / ODD = ODD
		ODD / EVEN = NEITHER	

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2 .. 50 \qquad U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2 + U\right)}{E}$$

$$B_{E,U} := B(E,U) \cdot (1 + \operatorname{floor}(B(E,U)) - \operatorname{ceil}(B(E,U)))$$

U

E

B =

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	0	6	9	0	17	22	0	34	41	0	57	66	0	86	97
4	0	4	6	0	0	15	19	0	0	34	40	0	0	61	69	0
5	3	0	0	0	11	14	0	0	0	30	35	0	0	0	59	66
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	4	0	0	0	0	0	16	19	0	0	0	0	0	43	48	0
8	0	0	0	9	11	0	0	0	0	0	0	32	36	0	0	0
9	5	0	0	0	0	0	0	0	21	24	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	6	0	0	0	0	0	0	0	0	0	26	29	0	0	0	0
12	0	0	9	0	0	14	16	0	0	23	0	0	0	0	38	0
13	7	0	0	0	0	0	0	0	0	0	0	0	31	34	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	8	0	0	0	0	15	0	0	0	23	0	0	0	0	36	39
16	0	0	0	0	0	0	0	19	21	0	0	0	0	0	0	0
17	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1,2..50$ $U := 0,1..90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2+U\right)}{E}$$

$$B_{E,U} := B(E,U)\cdot(1+\mathrm{floor}(B(E,U)) - \mathrm{ceil}(B(E,U)))$$

U

E

B =

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	256	289	324	361	400	441	484	529	576	625	676	729	784	841	900	961
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	97	0	121	134	0	162	177	0	209	226	0	262	281	0	321	342
4	0	0	96	106	0	0	139	151	0	0	190	204	0	0	249	265
5	66	0	0	0	98	107	0	0	0	147	158	0	0	0	206	219
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	84	91	0	0	0	0	0	139	148	0	0
8	0	0	0	0	71	77	0	0	0	0	0	0	126	134	0	0
9	0	0	56	61	0	0	0	0	0	0	0	109	116	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	69	74	0	0	0	0	0	0	0	0
12	0	0	49	53	0	0	66	0	0	0	0	91	0	0	108	114
13	0	0	0	0	0	0	0	0	0	0	82	87	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	39	0	0	0	0	56	0	0	0	72	0	0	0	0	95	100
16	0	0	0	0	0	0	0	0	66	70	0	0	0	0	0	0
17	0	41	44	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	46	49	0	0	0	0	0	0	0	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2..50 \qquad U := 0, 1..90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2+U\right)}{E}$$

$$B_{E,U} := B(E,U)\cdot(1 + \mathrm{floor}(B(E,U)) - \mathrm{ceil}(B(E,U)))$$

U

E

B =

	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	961	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	342	0	386	409	0	457	482	0	534	561	0	617	646	0	706	737
4	265	0	0	316	334	0	0	391	411	0	0	474	496	0	0	565
5	219	0	0	0	275	290	0	0	0	354	371	0	0	0	443	462
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	208	219	0	0	0	0	0	291	304	0	0	0
8	0	0	0	0	0	197	207	0	0	0	0	0	0	284	296	0
9	0	0	0	0	0	180	189	0	0	0	0	0	0	0	269	280
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	134	141	0	0	0	0	0	0	0	0	0	221	230	0
12	114	0	0	133	0	0	0	0	168	0	0	191	199	0	0	224
13	0	0	0	0	0	0	0	0	159	166	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	100	0	0	0	0	127	0	0	0	151	0	0	0	0	184	191
16	0	0	0	0	0	0	0	0	0	145	151	0	0	0	0	0
17	0	0	0	108	113	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	121	126	0	0	0	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2 .. 50$ $U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2 + U\right)}{E}$$

$$B_{E,U} := B(E,U) \cdot (1 + \text{floor}(B(E,U)) - \text{ceil}(B(E,U)))$$

U

E

B =

	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	737	0	801	834	0	902	937	0	$\cdot 10^3$	$\cdot 10^3$	0	$\cdot 10^3$	$\cdot 10^3$	0	$\cdot 10^3$	$\cdot 10^3$
4	565	589	0	0	664	690	0	0	771	799	0	0	886	916	0	0
5	462	0	0	0	542	563	0	0	0	651	674	0	0	0	770	795
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	388	403	0	0	0	0	0	499	516	0	0	0	0
8	0	0	0	0	0	0	387	401	0	0	0	0	0	0	506	522
9	280	0	0	0	0	0	0	0	376	389	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	330	341	0	0	0	0	0
12	224	0	0	0	0	269	0	0	298	308	0	0	339	0	0	0
13	0	0	0	0	0	0	262	271	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	191	0	0	0	0	228	0	0	0	260	0	0	0	0	303	312
16	0	0	0	0	0	0	0	0	0	0	256	264	0	0	0	0
17	0	0	0	0	0	209	216	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	234	241	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2 .. 50 \qquad U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2+U\right)}{E}$$

$$B_{E,U} := B(E,U)\cdot(1 + \mathrm{floor}(B(E,U)) - \mathrm{ceil}(B(E,U)))$$

U

E

B =

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	8	0	0	0	0	15	0	0	0	23	0	0	0	0	36	39
16	0	0	0	0	0	0	0	19	21	0	0	0	0	0	0	0
17	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	17	0	0	0	24	26	0	0	0	35	0
21	11	0	0	0	0	0	19	0	0	0	0	0	0	0	35	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	16	0	0	0	0	0	0	0	29	31	0	0	0
25	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	22	0	0	0	0	0	0	34	36	0
29	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	39
33	17	0	0	0	0	0	0	0	0	0	0	32	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2 .. 50 \qquad U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2 + U\right)}{E}$$

$$B_{E,U} := B(E,U) \cdot (1 + \text{floor}(B(E,U)) - \text{ceil}(B(E,U)))$$

U

E

B =

	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	36	39	0	0	0	0	56	0	0	0	72	0	0	0	0	95
16	0	0	0	0	0	0	0	0	0	66	70	0	0	0	0	0
17	0	0	41	44	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	46	49	0	0	0	0	0	0	0	0	0	0
20	35	0	0	0	0	0	0	0	0	0	0	68	0	0	0	83
21	35	0	0	0	0	0	51	54	0	0	0	0	0	74	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	56	59	0	0	0	0	0	0
24	0	0	0	0	0	0	50	0	0	0	0	0	0	71	0	0
25	0	0	0	0	0	0	0	0	0	0	61	64	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	66	69	0	0
28	36	0	0	0	0	0	0	52	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	71	74
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	39	41	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	52	0	0	0	0	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2 .. 50 \qquad U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2 + U\right)}{E}$$

$$B_{E,U} := B(E,U) \cdot (1 + \mathrm{floor}(B(E,U)) - \mathrm{ceil}(B(E,U)))$$

U

E

B =

	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	95	100	0	0	0	0	127	0	0	0	151	0	0	0	0	184
16	0	0	0	0	0	0	0	0	0	0	145	151	0	0	0	0
17	0	0	0	0	108	113	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	121	126	0	0	0	0	0	0
20	83	87	0	0	0	104	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	106	0	0	0	0	0	134	139	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	100	104	0	0	0	0	0	0	0	139
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	91	0	0	0	0	0	0	117	121	0	0
29	74	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	76	79	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	81	84	0	0	0	0	0	0	0	0	0	0	121

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1,2..50 \qquad U := 0,1..90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2+U\right)}{E}$$

$$B_{E,U} := B(E,U)\cdot(1+\mathrm{floor}(B(E,U)) - \mathrm{ceil}(B(E,U)))$$

U

E

B =

	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	184	191	0	0	0	0	228	0	0	0	260	0	0	0	0	303
16	0	0	0	0	0	0	0	0	0	0	0	256	264	0	0	0
17	0	0	0	0	0	0	209	216	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	234	241	0	0
20	0	159	0	0	0	182	188	0	0	0	213	0	0	0	0	0
21	0	0	0	0	171	0	0	0	0	0	0	0	219	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	147	152	0	0	0	0	0	0	0	0	0	0	0	0	0
24	139	0	0	0	0	0	0	174	0	0	0	0	0	0	0	219
25	0	0	0	0	0	160	165	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	173	178	0	0	0	0	0
28	0	0	0	0	0	151	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	186	191	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	134	138	0	0	0	0	0	0	0	0	0	0	0
33	121	0	0	0	0	0	0	0	0	0	161	0	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL.MCD

$E := 1, 2 .. 50$ $U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2 + U\right)}{E}$$

$$B_{E,U} := B(E,U)\cdot(1 + \text{floor}(B(E,U)) - \text{ceil}(B(E,U)))$$

U

E

B =

	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	303	312	0	0	0	0	359	0	0	0	399	0	0	0	0	452
16	0	0	0	0	0	0	0	0	0	0	0	0	399	409	0	0
17	0	0	0	0	0	0	0	0	344	353	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	290	0	0	0	321	329	0	0	0	362
21	0	0	0	259	266	0	0	0	0	0	310	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	284	291	0	0	0	0	0
24	219	225	0	0	0	0	0	0	0	276	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	309
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	216	0	0	0	0	0	0	256	262	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	199	204	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	212	217	0	0	0	0	0	0	0	0

FULCRUM NUMBERS AS FUNCTION OF E and U JULY 21, 2006
TABFUL MCD

$E := 1, 2 .. 50 \qquad U := 0, 1 .. 90$

$$B(E,U) := U + \frac{(E+1)}{2} + \frac{\left(U^2 + U\right)}{E}$$

$$B_{E,U} := B(E,U) \cdot (1 + \text{floor}(B(E,U)) - \text{ceil}(B(E,U)))$$

U

E

B =

	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	452	463	0	0	0	0	520	0	0	0	568	0	0	0	0	631
16	0	0	0	0	0	0	0	0	0	0	0	0	0	574	586	0
17	0	0	0	0	0	0	0	0	0	0	513	524	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	385	394	0	0	0	0	0	0	0	0	0	0	0	0	0
20	362	0	0	0	0	0	0	0	0	0	0	461	0	0	0	500
21	0	0	0	374	0	0	0	0	0	426	435	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	325	0	0	0	0	0	0	0	386	394	0	0	0	0	0
25	309	316	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	334	341	0	0	0	0	0	0	0	0
28	0	0	0	306	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	359	366	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	293	299	0	0	0	0	0	0	0	0	0
33	0	0	0	276	0	0	0	0	0	0	0	0	0	336	0	0

FULCRUMS

$\exists$ 2 kinds of Fulcrum Numbers

1) Those at center <u>between</u> two equal sums

2) Those at center <u>participating</u> in the equal sums

NON NUMBER FULCRUM

$\Sigma_1 \mid \Sigma_2$

$\cdots n \mid n+1 \ldots$

Example

Ⓘ $5 = 2\ 3\ ④\ 5$

$15 = 1\ 2\ 3\ 4\ 5\ ⑥\ 7\ 8 = 15$

$21 = 6\ 7\ 8\ ⑨\ 10\ 11 = 21$

(II) $9 = 2\ 3\ ④\ 5 \ = 9$

$21 = 1\ 2\ 3\ 4\ 5\ ⑥\ 7\ 8 = 21$

$30 = 6\ 7\ 8\ ⑨\ 10\ 11 = 30$

All type I are also type II

because $\Sigma_v \quad n \quad \Sigma_H$ when $\Sigma_v = \Sigma_H$

$\downarrow$

$\Sigma_v + n \qquad \underset{H}{\Sigma_c} + n$

and

$\Sigma_v - n \qquad \Sigma_H - n$

all type II are also type I

Question; as in the example above

I-⑨ sums = II-⑥ sums

how often does this occur?

$54 = 12\ 13\ 14\ 15\ (16)\ 17\ 18\ 19 = 54$

$70 \qquad +16 \qquad +16 \qquad = 70$

NOTATION CHANGE
E → P
U → K

Alternative Demonstration

5) $B-U = \frac{U(U+1)}{E} + \frac{E+1}{2}$

In general, writing equation $\overset{5}{3}$b) for values of $\overset{E}{\text{P}}$ from 1 to 10,

$\overset{E}{P = 1}$	$B = \overset{U^2}{K^2} + 2\overset{U}{K} + 1$	All integers $\overset{U}{K}$ give an integer value for B, as above
$P = 2$	$2B = K^2 + 3K + 3$	No integer values for B are possible, as above
$P = 3$	$3B = K^2 + 4K + 6$	
$P = 4$	$4B = K^2 + 5K + 10$	
$P = 5$	$5B = K^2 + 6K + 15$	
$P = 6$	$6B = K^2 + 7K + 21$	
$P = 7$	$7B = K^2 + 8K + 28$	
$P = 8$	$8B = K^2 + 9K + 36$	
$P = 9$	$9B = K^2 + 10K + 45$	
$P = 10$	$10B = K^2 + 11K + 55$	
----	-------------------	
$P = P$	$PB = K^2 + (P + 1)K + P(P+1)/2$	3c) $EB = U^2 + (E+1)U + \frac{E(E+1)}{2}$

There are four cases: P even, K even
P even, K odd
P odd, K even
P odd, K odd

~~The case P even/K even~~

The case P is odd

In equation **3c)**, Set $\overset{U = E}{K = P}$, then

$B = \frac{5P + 3}{2}$ and B will be an integer for any value of P that is odd.

$B = \frac{5E+3}{2}$ $B = F(E)$ E, N

Set K = P - 1, then

$B = \frac{5P - 3}{2}$ and B will be an integer for any value of P that is odd.

Set K = 2P, then $B = \frac{13P + 5}{2}$ Set K = 2P - 1, then $B = \frac{13P - 5}{2}$

Setting K = 3P or 3P-1 respectively gives $B = \frac{25P \pm 7}{2}$

Setting K = 4P or 4P-1 respectively gives $B = \frac{41P \pm 9}{2}$

Setting K = 5P or 5P-1 respectively gives $B = \frac{61P \pm 11}{2}$

(

✓ General

In general for K = NP or NP-1

3d) $$B = \frac{(2N^2 + 2N + 1)\overset{E}{P} \pm (2N + 1)}{2}$$ 2 sequences ±

E

It is evident from equation **3d)** that for P odd and any integer value of N, B will be an integer. This function B = B(P,N) gives all the fulcrum numbers possible for odd P, [P = 1,3,5,7,...] using any positive integer N.

$\frac{odd \cdot odd \pm odd}{2} = \frac{even}{2}$

The case P is even

Referring to equation $\overset{5}{\text{3b}}$), the numerator $\overset{U(V+1)}{K(K+1)}$ will always be even whether $\overset{V}{K}$ is even or odd. And in the case where P also is even, it is useful to write

$K^2+K = 2^n \times T$ where T is odd and $P = 2^d \times G$ where G is odd, This gives.

$$\frac{K^2 + K}{P} = \frac{2^n\ T}{2^d\ G} = \frac{T}{2^{d-n}\ G}$$

When n = d, T and G both being odd, when divisible, the quotient Q, will also be odd. Hence, the numerator in

$$\frac{2Q + P + 1}{2}$$

4, 8, 12, 16

will be odd and there will be no integer solutions to **3b)**

5

The case P even (continued)

Next, consider the case n > d. The numerator will be even and the denominator will be odd. The quotient will always be even and added to (P + 1)/2, as before, an odd numerator allows no integral solutions.

This restricts the possibilities for integral solutions to those cases where d > n.

If d = n + 1, equation **3b)** takes the form

$$B - K = \frac{T}{2\,G} + \frac{2\,G + 1}{2}$$

This equation will have integer solutions whenever T is divisible by G. since the quotient Q (odd divided by odd) will always be odd. And

$$B - K = \frac{1}{2}\,(Q + 2G + 1)$$

The numerator is even and therefore there are integer solutions.

[However, values of $d > n + 1$ do not appear to facilitate solutions.--not proven]

Additional P even conclusions:

There can be no solutions of ~~3~~5**b)** when $P/2 = G$ is odd.

$$B - K = \frac{K(K+1)}{2G} + \frac{2G+1}{2} = \frac{1}{2}\,[\,K(K+1)/G + 2G+1\,]$$

K(K+1)/G when divisible will be even, therefore the quantity in square brackets will be odd, precluding any integer values for B. Thus there are no solutions of ~~3~~5b) for P = 2,6,10,14,18,...

FULCRUM NUMBERS

E
L − U   L/U

#	B	Σ	L	U	E P	ρ SEQUENCE δ P	
1	4	5	2	1	1	2	□
2	6	15	5	2	3	5/2	Rom
3	9	21	3	2	1	3/2	□ 1
	9	33	6	3	3	2	1 ∘
4	11	54 °	9	4	5	9/4	
5	14	85	10	5	5	2	1 ∘
6	15	90	9	5	4	9/5	
7	16	54 °	4	3	1	4/3	□ 1
	16	117	13	6	7	13/6	
8	17	~~308~~ 100°	8	5	3	8/3	1
9	19	135	10	6	4	5/3	1
10	21	204	17	8	9	17/8	
11	22	153	9	6	3	3/2	1 ∘
12	24	252	21	9	12	7/3	
	24	261	18	9	9	2	1 ∘
13	25	110	5	4	1	5/4	□ 1
14	26	315 °	21	10	11	21/10	
15	29	385	22	11	11	2	1 ∘
16	30	315 °	14	9	5	14/9	1
17	31	450	25	12	13	25/12	
18	32	418	19	11	8	19/11	
19	34	308 °	11	8	3	11/8	1
17	34	351	13	9	4	13/9	1
	34	533	26	13	13	2	1 ∘
25 missing 9	19	85	14	7	7	2	1 ∘∘

7
2 190

p1

FULCRUM NUMBERS

L-U L/U

#	B	Σ	L	~~R~~ U	~~S~~ E	ρ	SEQUENCE ∂ ρ
20	35	405	15	10	5	3/2	[illegible] ı ∘
	35	595	34	14	20	17/7	RAM
21	36	195	6	5	1	6/5	□ ı
	36	510	20	12	8	5/3	ı ∘
	36	609	29	14	15	29/14	
22	38	637	26	14	12	13/7	ı
23	39	705	30	15	15	2	ı ∘
24	40	455	14	10	4	7/5	ı
25	41	414	12	9	3	4/3	ı ∘
	41	792	33	16	17	33/16	
26	43	650	20	13	7	13/7	ı ∘
27	44	901	34	17	17	2	ı ∘
28	46	999	37	18	19	37/18	
27	48	777	21	14	7	3/2	ı ∘
28	49	315	7	6	1	7/6	□ ı
	49	986	29	17	12	29/17	ı
	49	1121	38	19	19	2	∘
29	50	1210	44	20	24	11/5	
30	51	1230	41	20	21	41/20	
31	52	1323	49	21	28	7/3	∘
32	53	1125	30	18	12	5/3	ı ∘
33	54	1365	42	21	21	2	ı ∘
34	56	1105	26	17	9	26/17	ı
	56	1330	35	20	15	7/4	ı
	56	1485	45	22	23	45/22	

50 [illegible]

FULCRUM NUMBERS

#	B	Σ	L	~~K~~ U	~~P~~ δ	ρ SEQUENCE	5 ρ
35	57	693	14	11	3	14/11	1
36	59	931	19	14	5	19/14	1
	59	1633	46	23	23	2	o
37	61	884	17	13	4	17/13	1
	61	1269	27	18	9	3/2	1 o
	61	1764	49	24	25	49/24	
38	64	476	8	7	1	8/7	□ 1
	64	1925	50	25	25	2	o
	64	2015	62	26	36	31/13	
39	66	870	15	12	3	5/4	1 o
	66	1110	20	15	5	4/3	1 o
	66	1617	33	21	12	33/21	1
	66	1794	39	23	16	39/23	
	66	2067	53	26	27	53/26	
40	68	2025	45	25	20	9/5	1 o
41	69	1071	18	14	4	9/7	1
	69	1680	32	21	11	32/21	1
	69	2241	54	27	27	2	o
42	70	1980	40	24	16	5/3	o
43	71	1539	27	19	8	27/19	1
	71	2295	51	27	24	17/9	
	71	2394	57	28	29	57/28	
73 44.	72	2026	39	24	15	39/24	1

FULCRUM NUMBERS

	#	B	Σ	L	~~K~~ U	~~P~~ δ	ρ SEQUENCE	∂ ρ
	45	74	1881	33	22	11	3/2	1 0
		74	2376	48	27	21	16/9	1
		74	2581	58	29	29	2	1 0
	46	76	2745	61	30	31	61/30	
	47	77	1750	28	20	8	7/5	1 0
	48	79	2945	62	31	31	2	0
	49	81	684	9	8	1	9/8	□ 1
		81	3120	65	32	33	65/32	
		81	3234	77	33	44	7/3	0
	50	82	2375	38	25	13	38/25	1
	51	83	2842	49	29	20	49/29	1
	52	84	1890	27	20	7	27/20	1
		84	3333	66	33	33	2	1 0
	53	86	1309	17	14	3	17/14	1
		86	3519	69	34	35	69/34	
		86	3640	80	35	45	16/7	
	54	87	2613	39	26	13	3/2	1 0
		87	3075	50	30	20	5/3	1 0
		87	3675	75	35	40	15/7	
	55	89	3745	70	35	35	2	0
	56	91	3142	28	21	7	4/3	1 0
		91	~~3589~~ 2717	~~62~~ 38	~~34~~ 26	~~28~~ 12	19/13 13	1
		91	3689	62	34	28	31/17	1
~~97~~		91	3942	73	36	37	73/36	
98	57	93	4275	90	38	52	45/19	

P 4

FULCRUM NUMBERS

#	B	Σ	L	~~K~~ U	~~P~~ δ	ρ SEQUENCE	δ	ρ
58	94	4181	74	37	37	2	1	o
59	95	3190	44	29	15	44/29	1	
60	96	1785	21	17	4	21/17	1	
	96	4389	77	38	39	77/38		
	96	4524	87	39	48	87/39		
61	97	1575	18	15	3	6/5	1	o
62	98	2052	24	19	5	24/19	1	
63	99	4641	78	39	39	2	1	o
64	100	3465	45	30	15	3/2	1	o
	100	4130	59	35	24	59/35	1	
65	101	4860	81	40	41	81/40		
66	104	4131	54	34	20	27/17	1	
	104	4410	60	36	24	5/3	1	o
	104	5125	82	41	41	2		
67	106	2079	22	18	4	11/9	1	
	106	4340	56	35	21	8/5	1	o
	106	5355	85	42	43	85/42		
68	107	2350	25	20	5	5/4	1	o
69	108	3567	41	29	12	41/29	1	
	108	4125	50	33	17	50/33		
	108	5742	99	44	55	9/4		o
70	109	3185	35	26	9	35/26	1	
	109	5633	86	43	43	2		o
71	110	5985	105	45	60	21/9		

100^2 □

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SEQUENCES E $Q = \frac{L}{V}$

P

L V J-K FULCRUM NUMBERS : B

d v

Page 1

	#	B	J	K	P	S	#	B	J	K	P	S	
② ③	1	4	2	1	1	5	25	35	15	10	5	405	④
① ⑤	2	6	5	2	3	15	26	35	34	14	20	595	①
② ④	3	9	3	2	1	21	27	36	6	5	1	195	②
③	4	9	6	3	3	33	28	36	20	12	8	510	⑥
⑤	5	11	9	4	5	54	29	36	29	14	15	609	⑤
③	6	14	10	5	5	85	30	38	26	14	12	637	
⑩	7	15	9	5	4	90	31	39	30	15	15	705	③
②	8	16	4	3	1	54	32	40	14	10	4	455	12
⑤ ⑦	9	16	13	6	7	117	33	41	33	16	17	792	20 4414 ⑤
⑨	10	17	8	5	3	308	34	41 43	12 20	9 13	3 7	414 650	⑦ ⑨
⑥	11	19	10	6	4	135	35	44	34	17	17	901	③
⑤	12	21	17	8	9	204	36	46	37	18	19	999	⑧
④	13	22	9	6	3	153	27	48	21	14	7	777	④
⑧	14	23	21	9	12	252	28	49	7	6	1	315	②
③	15	24	18	9	9	261	39	49	29	17	12	986	⑩
②	16	25	5	4	1	110	40	49	38	19	19	1121	③
⑤	17	26	21	10	11	315	41	50	44	20	24	1210	
③	18	29	22	11	11	385	42	51	41	20	21	1230	⑤
⑨	19	30	14	9	5	315	43	52	49	21	28	1323	⑧
⑤	20	31	25	12	13	450	44	53	30	18	12	1125	⑥
⑩	21	32	19	11	8	418	45	54	42	21	21	1365	③
MAY Δ=12 ⑪	22	34	11	8	3	308	46	56	26	17	9	1105	⑨
	23	34	13	9	4	351	47	56	35	20	15	1330	
③	24	34	26	13	13	533	48	56	45	22	23	1485	⑤

↓
49

① Ramanujan
② Square
③ L/U = 2/1
④ L/U = 3/2
⑤ ~~L/U = 5/2~~ odd δ
⑥ L/U 5/3
⑦
⑧ L/U 7/3
⑨
⑩

36

FULCRUM NUMBERS B page 2

add a 61 ④ 27-18-9

	#	B	J	K	P	S	#	B	J	K	P	S	
	50	57	14	11	3	693	74	74	58	29	29	2581	③ p = 0
⑪	51	59	18	14	5	931	75	76	61	30	31	2945	
③	52	59	48	23	23	1633	76	77	28	20	8	1750	
	53	61	17	13	4	884	77	79	62	31	31	2945	③
③	54	61	49	24	25	1764	78	81	9	8	1	684	②
②	55	64	8	7	1	476	79	81	65	32	33	3120	⑤
③	56	64	50	25	25	1925	80	81	77	33	44	3234	
	57	64	62	26	36	2015	81	82	38	25	13	2375	
	58	66	15	12	3	870	82	83	49	29	20	2842	
⑦	59	66	20	15	5	1110	83	84	27	20	7	1890	
	60	66	33	21	12	1617	84	84	66	33	33	3333	③
⑩	61	66	39	23	16	1794	85	86	17	14	3	1309	
⑤	62	66	53	26	27	2067	86	86	69	34	35	3519	⑤
	63	68	45	25	20	2025	87	86	80	35	45	3640	
	64	69	18	14	4	1071	88	87	39	26	13	2613	④
	65	69	32	21	11	1680	89	87	50	30	20	3075	⑥
③	66	69	54	27	27	2241	90	87	75	35	40	3675	
⑥	67	70	40	24	16	1980	91	89	70	35	35	3745	③
	68	71	27	19	8	1539	92	91	28	21	7	2142	⑦
	69	71	67	27	28	2295	93	91	38	26	12	2717	
⑤	70	71	57	28	29	2394	94	91	62	34	28	3689	
	71	72	39	24	15	2026	95	91	73	36	37	3942	⑤
④	72	74	33	22	11	1881	96	93	90	38	52	4275	
	73	74	48	27	27	2376	97	94	74	37	37	4181	③
④		61	27	18	9	1269		60					

24
72
9

47

23
49
2

FULCRUM NUMBERS B   page 3

#	B	J	K	P	S	#	B	J	K	P	S
98	95	44	29	15	3190						
99	96	21	17	4	1785						
100	96	77	38	39	4389 ~~4644~~						
101	96	87	38	38	4524						
102	97	18	15	3	1575						
103	98	24	19	5	2052						
③ 104	99	78	39	39	4641						
105	100	45	30	15	3465						
106	100	59	35	24	4130						
107	101	81	40	41	4860						
108	104	54	34	20	4131						
109	104	60	36	24	4410						
③ 110	104	82	41	41	5125						
111	106	22	18	4	2079						
112	106	56	35	21	4340						
113	106	85	42	43	5355						
114	107	25	20	5	2350						
115	108	41	29	12	3567						
116	108	50	33	17	4125						
117	108	99	44	55	5742						
118	109	35	26	9	3185						
③ 119	109	86	43	43	5633						
120	110	105	45	60	5985						
121											

~~102~~ ? nyet

98
24
2

106 ?

106	22	18	4	✓
	56	35	21	✓
	85	42	43	✓

FULNUM04.WPD October 7, 2003

FULCRUM NUMBER FILES ~~ROOT FILES~~

FULPKB0Z.MCD	VERT=P,	HORIZ=K,	TABLE=B	GREEN
FULKBP0Z.MCD	VERT=K,	HORIZ=B,	TABLE=P	GOLD
FULPBK0Z.MCD	VERT=P,	HORIZ=B,	TABLE=K	YELLOW
FULPKS0Z.MCD	VERT=P,	HORIZ=K,	TABLE=Σ	SALMON

TABLES OF FULCRUM NUMBERS

L−U (δ) $\overline{\quad\quad}$

U

✓

FULPKBOZ

~~FULCRUMY~~ 03-09-30

VERTICAL = ~~B~~ E
HORIZONTAL = ~~K~~ U
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

B = a fulcrum number

K = U
B = L−U {δ}

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

5) $B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$

W(E,U)

(D) $U_{P,K} := (-(\text{ceil}(W(P,K)) - \text{floor}(W(P,K)) - 1))\cdot W(P,K)$

(F − c)

B > L = U + E

B > L

B > J

B > K + P = L

B < ~~δ+U~~ are meaningless

Ram

U

K = U

P

δ

E−U

E

(D) U =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	6	9	0	17	22	0	34	41	0	57	66	0	86	97	0
4	4	6	0	0	15	19	0	0	34	40	0	0	61	69	0	0
5	0	0	0	11	14	0	0	0	30	35	0	0	0	59	66	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	16	19	0	0	0	0	0	43	48	0	0
8	0	0	9	11	0	0	0	0	0	0	32	36	0	0	0	0
9	0	0	0	0	0	0	0	21	24	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	26	29	0	0	0	0	0
12	0	9	0	0	14	16	0	0	23	0	0	0	0	38	0	0
13	0	0	0	0	0	0	0	0	0	0	0	31	34	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	15	0	0	0	23	0	0	0	0	36	39	0
16	0	0	0	0	0	0	19	21	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	41
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	17	0	0	0	24	26	0	0	0	35	0	0

square no 2s

Ram

Ceil(x) = smallest integer ≥ x B

floor(x) = greatest integer ≤ x B

$[\text{ceil}(x) - \text{floor}(x)]$

$\{-[(B+1) - (B-1)] - 1\}\cdot B$

−2 −1

−3

$\{-(\text{floor} - \text{ceil}) - 1\}\cdot W$

FULPRB0Z

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	289	324	361	400	441	484	529	576	625	676	729	784	841	900	961	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	121	134	0	162	177	0	209	226	0	262	281	0	321	342	0
4	0	96	106	0	0	139	151	0	0	190	204	0	0	249	265	0
5	0	0	0	98	107	0	0	0	147	158	0	0	0	206	219	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	84	91	0	0	0	0	0	139	148	0	0	0
8	0	0	0	71	77	0	0	0	0	0	0	126	134	0	0	0
9	0	56	61	0	0	0	0	0	0	0	109	116	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	69	74	0	0	0	0	0	0	0	0	0
12	0	49	53	0	0	66	0	0	0	0	91	0	0	108	114	0
13	0	0	0	0	0	0	0	0	0	82	87	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	56	0	0	0	72	0	0	0	0	95	100	0
16	0	0	0	0	0	0	0	66	70	0	0	0	0	0	0	0
17	41	44	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	46	49	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	68	0	0	0	83	87	0

✓

FULPROOZ

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B \Rightarrow W(P,K) := K + \frac{\left(K^2 + K\right)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
1	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	386	409	0	457	482	0	534	561	0	617	646	0	706	737	0
4	0	0	316	334	0	0	391	411	0	0	474	496	0	0	565	589
5	0	0	0	275	290	0	0	0	354	371	0	0	0	443	462	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	208	219	0	0	0	0	0	291	304	0	0	0	0
8	0	0	0	0	197	207	0	0	0	0	0	0	284	296	0	0
9	0	0	0	0	180	189	0	0	0	0	0	0	0	269	280	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	134	141	0	0	0	0	0	0	0	0	0	221	230	0	0
12	0	0	133	0	0	0	0	168	0	0	191	199	0	0	224	0
13	0	0	0	0	0	0	0	159	166	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	127	0	0	0	151	0	0	0	0	184	191	0
16	0	0	0	0	0	0	0	0	145	151	0	0	0	0	0	0
17	0	0	108	113	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	121	126	0	0	0	0	0	0	0	0
20	0	0	0	104	0	0	0	0	0	0	0	0	0	0	159	0

✓

FULPRB03

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
1	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	801	834	0	902	937	0	$\cdot 10^3$	$\cdot 10^3$	0	$\cdot 10^3$	$\cdot 10^3$	0	$\cdot 10^3$	$\cdot 10^3$	0
4	589	0	0	664	690	0	0	771	799	0	0	886	916	0	0	$\cdot 10^3$
5	0	0	0	542	563	0	0	0	651	674	0	0	0	770	795	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	388	403	0	0	0	0	0	499	516	0	0	0	0	0
8	0	0	0	0	0	387	401	0	0	0	0	0	0	506	522	0
9	0	0	0	0	0	0	0	376	389	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	330	341	0	0	0	0	0	0
12	0	0	0	0	269	0	0	298	308	0	0	339	0	0	0	0
13	0	0	0	0	0	262	271	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	228	0	0	0	260	0	0	0	0	303	312	0
16	0	0	0	0	0	0	0	0	0	256	264	0	0	0	0	0
17	0	0	0	0	209	216	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	234	241	0	0	0	0
20	0	0	0	182	188	0	0	0	213	0	0	0	0	0	0	0

✓

FULPKB03

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P, K) := K + \frac{\left(K^2 + K\right)}{P} + \frac{(P + 1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
20	0	0	0	0	17	0	0	0	24	26	0	0	0	35	0	0
21	0	0	0	0	0	19	0	0	0	0	0	0	0	35	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	16	0	0	0	0	0	0	0	29	31	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	22	0	0	0	0	0	0	34	36	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	39	41
33	0	0	0	0	0	0	0	0	0	0	32	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	38	0	0
36	0	0	0	0	0	0	0	0	30	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	36	0	0	0	0

✓

FULPKB02

~~FULCRUM~~Y 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
20	0	0	0	0	0	0	0	0	0	68	0	0	0	83	87	0
21	0	0	0	0	51	54	0	0	0	0	0	74	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	56	59	0	0	0	0	0	0	0	0
24	0	0	0	0	50	0	0	0	0	0	0	71	0	0	0	0
25	0	0	0	0	0	0	0	0	61	64	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	66	69	0	0	0	0
28	0	0	0	0	0	52	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	71	74	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	76	79
32	41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	52	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	50	0	0	0	0	0	0	0	0	0	0	0
36	0	44	46	0	0	0	0	0	0	0	64	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	64	0	0	0	0	0

V

FULPRBOZ

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{\left(K^2 + K\right)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1))\cdot W(P,K)$$

B > K + P

K

P

U =

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
20	0	0	0	104	0	0	0	0	0	0	0	0	0	0	159	0
21	0	0	0	0	106	0	0	0	0	0	134	139	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	147	152
24	0	0	0	0	100	104	0	0	0	0	0	0	0	139	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	91	0	0	0	0	0	0	117	121	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	79	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	81	84	0	0	0	0	0	0	0	0	0	0	121	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	86	89	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	121	0
37	0	0	0	0	0	91	94	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	96	99	0	0	0	0	0	0	0

✓ nam

FULCRUMY 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\text{ceil}(W(P,K)) - \text{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
20	0	0	0	182	188	0	0	0	213	0	0	0	0	0	0	0
21	0	0	171	0	0	0	0	0	0	0	219	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	152	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	174	0	0	0	0	0	0	0	219	225	0
25	0	0	0	160	165	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	173	178	0	0	0	0	0	0	0
28	0	0	0	151	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	186	191	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	199
32	0	134	138	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	161	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	137	0	0	0	0	0	161	0	0	0	0	0	0
36	0	0	0	0	0	0	0	151	155	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	139	0	0	0	0	0	0	0	0	0	0

FULCHART ✓

FULCRUM NUMBER SEQUENCES
SEEDED BY PERFECT SQUARES
THE P = 1 SEQUENCES

$s := 1, 2 .. 10$

$n := 1, 2 .. 20$

$S = ^2,$

FORMULA II

$(s+1)$

$B_{s,n} := n \cdot s^2 + (n-1) \cdot (s+1)^2$

$B(s,n) = B_1(s) + (n-1)\Delta B(s)$

TABLE $B(s,n)$

$\Delta B(s) \pm 2s^2 + 2s + 1$

n

$B_1 = s^2$

$B(s,1)$

$B_1 = s^2$

s

B =

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	6	11	16	21	26	31	36	41	46	51	56
2	0	4	17	30	43	56	69	82	95	108	121	134	147
3	0	9	34	59	84	109	134	159	184	209	234	259	284
4	0	16	57	98	139	180	221	262	303	344	385	426	467
5	0	25	86	147	208	269	330	391	452	513	574	635	696
6	0	36	121	206	291	376	461	546	631	716	801	886	971
7	0	49	162	275	388	501	614	727	840	953	1066	1179	1292
8	0	64	209	354	499	644	789	934	1079	1224	1369	1514	1659
9	0	81	262	443	624	805	986	1167	1348	1529	1710	1891	2072
10	0	100	321	542	763	984	1205	1426	1647	1868	2089	2310	2531

ΔB	Δ^2	Δ^3
5		
13	8	4
25	12	4
41	16	4
61	20	4
85	24	4
113	28	4
145	32	4
181	36	4
221	40	

Square

$\delta = 3$

$\delta = 5$

All entries are B's

$s = ?$

$n = ?$

$\delta = 7$

$\delta = 9$

only one of the two sequences?

$\delta = 4$?

FULCHART ✓

FULCRUM NUMBER SEQUENCES
SEEDED BY PERFECT SQUARES
THE P = 1 SEQUENCES

$s := 1, 2 .. 10$

$n := 1, 2 .. 20$

FORMULA I

(s-1)

$$B_{s,n} := n \cdot s^2 + (n-1) \cdot (s-1)^2$$

TABLE: B(s, n)

$\Delta B = 2s^2 - 2s + 1$

n

B(s,1)
B=s²

s

B =

	0	1	2	3	4	5	6	7	8	9	10	11	12	ΔB	Δ²
0	0	0	0	0	0	0	0	0	0	0	0	0	0		
1	0	1	2	3	4	5	6	7	8	9	10	11	12	1	
2	0	4	9	14	19	24	29	34	39	44	49	54	59	5	4
3	0	9	22	35	48	61	74	87	100	113	126	139	152	13	8
4	0	16	41	66	91	116	141	166	191	216	241	266	291	25	12
5	0	25	66	107	148	189	230	271	312	353	394	435	476	41	16
6	0	36	97	158	219	280	341	402	463	524	585	646	707	61	20
7	0	49	134	219	304	389	474	559	644	729	814	899	984	85	24
8	0	64	177	290	403	516	629	742	855	968	1081	1194	1307	113	28
9	0	81	226	371	516	661	806	951	1096	1241	1386	1531	1676	145	32
10	0	100	281	462	643	824	1005	1186	1367	1548	1729	1910	2091	181	36

↑ square ↑ δ=3 ↑ δ=5 ↑ δ=7

FULCHART ✓

FULCRUM NUMBERS
DERIVED FROM
SQUARES

$s := 1, 2 .. 10$

$n := 1, 2 .. 20$

$B_{s,n} := n \cdot s^2 + (n-1) \cdot (s-1)^2$

n

s

B =

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	4	9	14	19	24	29	34	39	44	49	54	59
3	0	9	22	35	48	61	74	87	100	113	126	139	152
4	0	16	41	66	91	116	141	166	191	216	241	266	291
5	0	25	66	107	148	189	230	271	312	353	394	435	476
6	0	36	97	158	219	280	341	402	463	524	585	646	707
7	0	49	134	219	304	389	474	559	644	729	814	899	984
8	0	64	177	290	403	516	629	742	855	968	1081	1194	1307
9	0	81	226	371	516	661	806	951	1096	1241	1386	1531	1676
10	0	100	281	462	643	824	1005	1186	1367	1548	1729	1910	2091

FULCHART ✓

FULCRUM NUMBER SEQUENCES
SEEDED BY PERFECT SQUARES
THE P = 1 SEQUENCES

$s := 1, 2 .. 10$

$n := 1, 2 .. 20$

FORMULA II

$B_{s,n} := n \cdot s^2 + (n-1) \cdot (s+1)^2$

n

s

B =

	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	36	41	46	51	56	61	66	71	76	81	86	91	96
2	95	108	121	134	147	160	173	186	199	212	225	238	251
3	184	209	234	259	284	309	334	359	384	409	434	459	484
4	303	344	385	426	467	508	549	590	631	672	713	754	795
5	452	513	574	635	696	757	818	879	940	1001	1062	1123	1184
6	631	716	801	886	971	1056	1141	1226	1311	1396	1481	1566	1651
7	840	953	1066	1179	1292	1405	1518	1631	1744	1857	1970	2083	2196
8	1079	1224	1369	1514	1659	1804	1949	2094	2239	2384	2529	2674	2819
9	1348	1529	1710	1891	2072	2253	2434	2615	2796	2977	3158	3339	3520
10	1647	1868	2089	2310	2531	2752	2973	3194	3415	3636	3857	4078	4299

FULCHART ✓

FULCRUM NUMBER SEQUENCES
SEEDED BY PERFECT SQUARES
THE P = 1 SEQUENCES

$s := 1, 2 .. 10$

$n := 1, 2 .. 20$

FORMULA I

$$B_{s,n} := n \cdot s^2 + (n-1) \cdot (s-1)^2$$

n

s

B =

	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	8	9	10	11	12	13	14	15	16	17	18	19	20
2	39	44	49	54	59	64	69	74	79	84	89	94	99
3	100	113	126	139	152	165	178	191	204	217	230	243	256
4	191	216	241	266	291	316	341	366	391	416	441	466	491
5	312	353	394	435	476	517	558	599	640	681	722	763	804
6	463	524	585	646	707	768	829	890	951	1012	1073	1134	1195
7	644	729	814	899	984	1069	1154	1239	1324	1409	1494	1579	1664
8	855	968	1081	1194	1307	1420	1533	1646	1759	1872	1985	2098	2211
9	1096	1241	1386	1531	1676	1821	1966	2111	2256	2401	2546	2691	2836
10	1367	1548	1729	1910	2091	2272	2453	2634	2815	2996	3177	3358	3539

FULCHART

$s := 1, 2 .. 10$

$n := 1, 2 .. 20$

$B_{s,n} := n \cdot s^2 + (n-1) \cdot (s-1)^2$

n

s

B =

	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	8	9	10	11	12	13	14	15	16	17	18	19	20
2	39	44	49	54	59	64	69	74	79	84	89	94	99
3	100	113	126	139	152	165	178	191	204	217	230	243	256
4	191	216	241	266	291	316	341	366	391	416	441	466	491
5	312	353	394	435	476	517	558	599	640	681	722	763	804
6	463	524	585	646	707	768	829	890	951	1012	1073	1134	1195
7	644	729	814	899	984	1069	1154	1239	1324	1409	1494	1579	1664
8	855	968	1081	1194	1307	1420	1533	1646	1759	1872	1985	2098	2211
9	1096	1241	1386	1531	1676	1821	1966	2111	2256	2401	2546	2691	2836
10	1367	1548	1729	1910	2091	2272	2453	2634	2815	2996	3177	3358	3539

2

K B P
FUL~~BK~~P0Z.MCD 03-09-28

$B := 4, 5 .. 122$

$K := 1, 2 .. 50$

VERTICAL = K
HORIZONTAL = B
TABLE = P which means a B exists
with the corresponding value of K.

E(B,U)

$$P_{B,K} := \frac{1}{2}\cdot\left[(2\,(B-K)-1)-\sqrt{(2\,(B-K)-1)^2-8\,\left(K^2+K\right)}\right]$$

$$T_{B,K} := \mathrm{Re}\left(P_{B,K}\right)$$

$$U_{B,K} := \left(\mathrm{floor}\left(T_{B,K}\right)-\mathrm{ceil}\left(T_{B,K}\right)+1\right)\cdot\left(P_{B,K}\right)$$

$$W_{K,B} := U_{B,K}$$

E< B-U

$J = P + K$ $P < B - K$

L = E+U

B

K

U

W =

	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	3	0	0	0	0	0	0	1	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	5	4	0	3	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0	4	0	0	3	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

← B

↑
U

Table: δ

KBP
FUL~~BKP~~0Z.MCD 03-09-28

$B := 4, 5 .. 122$

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VERTICAL = K
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$$T_{B,K} := \mathrm{Re}\left(P_{B,K}\right)$$

$$U_{B,K} := \left(\mathrm{floor}\left(T_{B,K}\right)-\mathrm{ceil}\left(T_{B,K}\right)+1\right)\cdot\left(P_{B,K}\right)$$

$$W_{K,B} := U_{B,K}$$

J = P + K

P < B - K

B

K

W =

	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0
9	12	9	0	0	0	0	0	5	0	0	0	4	0	0	0	0	0	0	3	0
10	0	0	0	11	0	0	0	0	0	0	0	0	5	0	0	0	0	4	0	0
11	0	0	0	0	0	0	11	0	0	8	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	13	0	0	0	0	8	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	13	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	20	15	0	12	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	17	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

KBP

FUL~~BKP~~0Z.MCD 03-09-28

$B := 4, 5 .. 122$

$K := 1, 2 .. 50$

VERTICAL = K
HORIZONTAL = B
TABLE = P which means a B exists with the corresponding value of K.

$$P_{B,K} := \frac{1}{2} \cdot \left[(2\ (B-K)-1) - \sqrt{(2\ (B-K)-1)^2 - 8\ \left(K^2+K\right)} \right]$$

$$T_{B,K} := \mathrm{Re}\left(P_{B,K}\right)$$

$$U_{B,K} := \left(\mathrm{floor}\left(T_{B,K}\right) - \mathrm{ceil}\left(T_{B,K}\right) + 1\right) \cdot \left(P_{B,K}\right)$$

$$W_{K,B} := U_{B,K}$$

$J = P + K$

$P < B - K$

B

K

W =

	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
6	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
14	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	5	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	17	0	0	0	0	12	0	0	0	0	0	0	9	0	0	0	0	0
18	0	0	0	0	19	0	0	0	0	0	0	12	0	0	0	0	0	0	0	9
19	0	0	0	0	0	0	0	19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	24	21	0	0	0	0	15	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	28	0	21	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	23	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	23	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

KBP ✓

FUL~~BKP~~0Z.MCD 03-09-28

$B := 4, 5 .. 122$

$K := 1, 2 .. 50$

VERTICAL = K
HORIZONTAL = B
TABLE = P which means a B exists
with the corresponding value of K.

$$P_{B,K} := \frac{1}{2} \cdot \left[(2\,(B-K)-1) - \sqrt{(2\,(B-K)-1)^2 - 8\,\left(K^2+K\right)} \right]$$

$$T_{B,K} := \mathrm{Re}\left(P_{B,K}\right)$$

$$U_{B,K} := \left(\mathrm{floor}\left(T_{B,K}\right) - \mathrm{ceil}\left(T_{B,K}\right) + 1\right) \cdot \left(P_{B,K}\right)$$

$$W_{K,B} := U_{B,K}$$

J = P + K

P < B - K

B

K-7

K

W =

	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	0	0	0
21	0	0	0	0	0	12	0	0	11	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0	0	0	0
23	0	0	0	0	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	25	0	0	0	0	0	0	0	0	16	0	15	0	0	0	0	0	0	0	0
25	0	0	0	25	0	0	0	20	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	36	0	27	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	27	0	24	0	0	21	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	29	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	29	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	31	0	0	0	0

✓

$P := 1, 2 .. 70$ FULPBK0Z.MCD 03-09-26

$B := 1, 2 .. 120$

VERTICAL = P
HORIZONTAL = B
TABLE = K

$$W(P,B) := \frac{\left[-(1+P)+\sqrt{(1+P)^2+4\cdot\left[B\cdot P-\frac{\left(P^2+P\right)}{2}\right]}\right]}{2}$$

$$Q(P,B) := \mathrm{Re}(W(P,B))$$

$$U_{P,B} := (\mathrm{floor}(Q(P,B)) - \mathrm{ceil}(Q(P,B)) + 1)\cdot Q(P,B)$$

B

K + P < B

P

E

U =

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	3	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	-1	0	0	0	0	2	0	0	3	0	0	0	0	0	0	0	5	0	0
4	0	-2	0	0	1	0	2	0	0	0	0	0	0	0	0	5	0	0	0	6
5	0	-3	-1	0	0	0	0	0	0	0	0	4	0	0	5	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	-4	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	7
8	0	0	-4	0	0	0	0	0	0	3	0	4	0	0	0	0	0	0	0	0
9	0	-5	-5	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	-6	-6	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	-6	-3	0	0	0	0	2	0	0	0	0	5	0	6	0	0	0
13	0	-7	-7	-7	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	-8	-8	-8	-6	0	0	-1	0	0	0	0	0	0	0	5	0	0	0	0
16	0	0	0	0	-8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7
17	0	-9	-9	-9	-9	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	-10	-10	-10	-10	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0

✓

$P := 1, 2 .. 70$ FULPBK0Z.MCD 03-09-26

$B := 1, 2 .. 120$

VERTICAL = P
HORIZONTAL = B
TABLE = K

$$W(P,B) := \frac{\left[-(1+P)+\sqrt{(1+P)^2+4\cdot\left[B\cdot P-\frac{\left(P^2+P\right)}{2}\right]}\right]}{2}$$

$$Q(P,B) := \mathrm{Re}(W(P,B))$$

$$U_{P,B} := (\mathrm{floor}(Q(P,B)) - \mathrm{ceil}(Q(P,B)) + 1)\cdot Q(P,B)$$

B

K + P < B

P

U =

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	5	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	6	0	0	0	0	0	0	0	0	0	0	0	8	0	0	0	0
4	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	9	0	0	0	0	10	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0	12	0	0
9	0	0	8	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	10	0	0	11	0	0	0	0	0	0	0	0	0
12	0	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	14
13	0	0	0	0	0	0	0	0	0	0	0	0	12	0	0	13	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	14	0	0
16	7	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

20 14

$P := 1, 2 .. 70$ FULPBK0Z.MCD 03-09-26

$B := 1, 2 .. 120$

VERTICAL = P
HORIZONTAL = B
TABLE = K

$$W(P,B) := \frac{\left[-(1+P) + \sqrt{(1+P)^2 + 4\cdot\left[B\cdot P - \frac{\left(P^2+P\right)}{2}\right]}\right]}{2}$$

$Q(P,B) := \mathrm{Re}(W(P,B))$

$U_{P,B} := (\mathrm{floor}(Q(P,B)) - \mathrm{ceil}(Q(P,B)) + 1)\cdot Q(P,B)$

B

K + P < B

P

U =

	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
4	0	0	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	13	0	0	0	0	14	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	17	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	14	0	0	0	0	0	0	0	0	0	0	17	0	0	0	18	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	16	0	0	17	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	18	0	0	19	0	0	0	0	0	0	0	0

V

$P := 1, 2 .. 70$ FULPBK0Z.MCD 03-09-26

$B := 1, 2 .. 120$

VERTICAL = P
HORIZONTAL = B
TABLE = K

$$W(P,B) := \frac{\left[-(1+P)+\sqrt{(1+P)^2+4\cdot\left[B\cdot P-\frac{\left(P^2+P\right)}{2}\right]}\right]}{2}$$

$$Q(P,B) := \mathrm{Re}(W(P,B))$$

$$U_{P,B} := (\mathrm{floor}(Q(P,B)) - \mathrm{ceil}(Q(P,B)) + 1)\cdot Q(P,B)$$

K + P < B

B

P

U =

	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57
19	0	0	0	0	0	0	0	0	18	0	0	19	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	20	0	0	21	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	22	0
24	0	0	0	0	0	0	0	0	0	0	0	0	20	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	21	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	15	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	21	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	14	0	0	0	0	0	0	0	0	0	0	0	20	0	0	0	0	0	0	0
36	0	0	0	0	0	0	17	0	18	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

✓

$P := 1, 2 .. 70$ FULPBK0Z.MCD 03-09-26

$B := 1, 2 .. 120$

VERTICAL = P
HORIZONTAL = B
TABLE = K

$$W(P,B) := \frac{\left[-(1+P)+\sqrt{(1+P)^2+4\cdot\left[B\cdot P-\frac{\left(P^2+P\right)}{2}\right]}\right]}{2}$$

$$Q(P,B) := \mathrm{Re}(W(P,B))$$

$$U_{P,B} := (\mathrm{floor}(Q(P,B)) - \mathrm{ceil}(Q(P,B)) + 1)\cdot Q(P,B)$$

K + P < B

B

P

U =

	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	11	0	0	0	0	0	0	0	0	12	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	13	0	0	0	0	0	0	0	14	0	0	0	0	0	0	0
5	0	0	14	0	0	0	0	0	0	15	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	19	0	0	0	0	0
9	0	0	0	0	18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	21	0	0	0	0	22	0	0
12	0	0	0	0	0	0	0	0	0	21	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0	0	0
16	0	0	0	0	0	0	0	0	0	23	0	0	0	24	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$P := 1, 2 .. 70$

FULPBK0Z.MCD 03-09-26

$B := 1, 2 .. 120$

VERTICAL = P
HORIZONTAL = B
TABLE = K

$$W(P,B) := \frac{\left[-(1+P)+\sqrt{(1+P)^2+4\cdot\left[B\cdot P-\frac{\left(P^2+P\right)}{2}\right]}\right]}{2}$$

$$Q(P,B) := \mathrm{Re}(W(P,B))$$

$$U_{P,B} := (\mathrm{floor}(Q(P,B)) - \mathrm{ceil}(Q(P,B)) + 1)\cdot Q(P,B)$$

B

K + P < B

P

U =

	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	25	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	27	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	27	0	0	0	0	0
25	0	0	0	0	24	0	0	25	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	26	0	0	27	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	28	0	0	29	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	26	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FULPKS0Z.MCD 03-10-02

$P := 1, 2 .. 50$

$K := 1, 2 .. 40$

VERTICAL = P
HORIZONTAL = K
TABLE = S $\sum'\Delta$

$$B_{P,K} := K + \frac{(P+1)}{2} + \frac{\left(K^2 + K\right)}{P}$$

$$S_{P,K} := K \cdot \left(B_{P,K}\right) + K \cdot \frac{(K+1)}{2}$$

$$U_{P,K} := \left(\text{floor}\left(B_{P,K}\right) - \text{ceil}\left(B_{P,K}\right) + 1\right) \cdot S_{P,K}$$

K =0

P

8

E

$\sum'\Delta$

U =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	5	21	54	110	195	315	476	684	945	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	15	33	0	100	153	0	308	414	0	693	870	0	$\cdot 10^3$	$\cdot 10^3$	0
4	5	15	0	0	90	135	0	0	351	455	0	0	884	$\cdot 10^3$	0	0
5	0	0	0	54	85	0	0	0	315	405	0	0	0	931	$\cdot 10^3$	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	117	161	0	0	0	0	0	650	777	0	0
8	0	0	33	54	0	0	0	0	0	0	418	510	0	0	0	0
9	0	0	0	0	0	0	0	204	261	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	315	385	0	0	0	0	0
12	0	21	0	0	85	117	0	0	252	0	0	0	0	637	0	0
13	0	0	0	0	0	0	0	0	0	0	0	450	533	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	90	0	0	0	252	0	0	0	0	609	705	0
16	0	0	0	0	0	0	161	204	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	792
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	100	0	0	0	261	315	0	0	0	595	0	0

✓

FULPKS0Z.MCD 03-10-02

$P := 1, 2 .. 50$

$K := 1, 2 .. 40$

VERTICAL = P
HORIZONTAL = K
TABLE = S

$$B_{P,K} := K + \frac{(P+1)}{2} + \frac{\left(K^2 + K\right)}{P}$$

$$S_{P,K} := K \cdot \left(B_{P,K}\right) + K \cdot \frac{(K+1)}{2}$$

$$U_{P,K} := \left(\mathrm{floor}\left(B_{P,K}\right) - \mathrm{ceil}\left(B_{P,K}\right) + 1\right) \cdot S_{P,K}$$

K

P 14		15	16	17	18	19	20
	0	0	0	0	0	0	0
3255	1	$3.96 \cdot 10^3$	$4.76 \cdot 10^3$	$5.661 \cdot 10^3$	$6.669 \cdot 10^3$	$7.79 \cdot 10^3$	$9.03 \cdot 10^3$
	2	0	0	0	0	0	0
1309	3	$1.575 \cdot 10^3$	0	$2.21 \cdot 10^3$	$2.583 \cdot 10^3$	0	$3.45 \cdot 10^3$
1071	4	0	0	$1.785 \cdot 10^3$	$2.079 \cdot 10^3$	0	0
	5	$1.11 \cdot 10^3$	0	0	0	$2.052 \cdot 10^3$	$2.35 \cdot 10^3$
	6	0	0	0	0	0	0
	7	0	0	0	0	0	$1.89 \cdot 10^3$
	8	0	0	0	0	$1.539 \cdot 10^3$	$1.75 \cdot 10^3$
U =	9	0	0	$1.105 \cdot 10^3$	$1.269 \cdot 10^3$	0	0
	10	0	0	0	0	0	0
	11	0	0	0	0	0	0
	12	0	0	986	$1.125 \cdot 10^3$	0	0
	13	0	0	0	0	0	0
	14	0	0	0	0	0	0
	15	705	0	0	0	0	$1.33 \cdot 10^3$
	16	0	0	0	0	0	0
	17	0	792	901	0	0	0
	18	0	0	0	0	0	0
	19	0	0	0	999	$1.121 \cdot 10^3$	0

8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0
684	945	$1.265\cdot10^{3}$	$1.65\cdot10^{3}$	$2.106\cdot10^{3}$	$2.639\cdot10^{3}$	$3.255\cdot10^{3}$	$3.96\cdot10^{3}$
0	0	0	0	0	0	0	0
308	414	0	693	870	0	$1.309\cdot10^{3}$	$1.575\cdot10^{3}$
0	351	455	0	0	884	$1.071\cdot10^{3}$	0
0	315	405	0	0	0	931	$1.11\cdot10^{3}$
0	0	0	0	0	0	0	0
0	0	0	0	0	650	777	0
0	0	0	418	510	0	0	0
204	261	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	315	385	0	0	0	0
0	252	0	0	0	0	637	0
0	0	0	0	450	533	0	0
0	0	0	0	0	0	0	0
0	252	0	0	0	0	609	705
204	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

✓

FULPKS0Z.MCD 03-10-02

$P := 1, 2 .. 50$

$K := 1, 2 .. 40$

VERTICAL = P
HORIZONTAL = K
TABLE = S

$$B_{P,K} := K + \frac{(P+1)}{2} + \frac{\left(K^2 + K\right)}{P}$$

$$S_{P,K} := K \cdot \left(B_{P,K}\right) + K \cdot \frac{(K+1)}{2}$$

$$U_{P,K} := \left(\text{floor}\left(B_{P,K}\right) - \text{ceil}\left(B_{P,K}\right) + 1\right) \cdot S_{P,K}$$

K

P

U =

	20	21	22	23	24	25
0	0	0	0	0	0	0
1	$9.03\cdot10^{3}$	$1.0395\cdot10^{4}$	$1.1891\cdot10^{4}$	$1.3524\cdot10^{4}$	$1.53\cdot10^{4}$	$1.7225\cdot10^{4}$
2	0	0	0	0	0	0
3	$3.45\cdot10^{3}$	$3.948\cdot10^{3}$	0	$5.083\cdot10^{3}$	$5.724\cdot10^{3}$	0
4	0	$3.15\cdot10^{3}$	$3.575\cdot10^{3}$	0	0	$5.075\cdot10^{3}$
5	$2.35\cdot10^{3}$	0	0	0	$3.828\cdot10^{3}$	$4.275\cdot10^{3}$
6	0	0	0	0	0	0
7	$1.89\cdot10^{3}$	$2.142\cdot10^{3}$	0	0	0	0
8	$1.75\cdot10^{3}$	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	$1.68\cdot10^{3}$	$1.881\cdot10^{3}$	0	0	0
12	0	$1.617\cdot10^{3}$	0	0	0	0
13	0	0	0	0	0	$2.375\cdot10^{3}$
14	0	0	0	0	0	0
15	$1.33\cdot10^{3}$	0	0	0	$2.028\cdot10^{3}$	0
16	0	0	0	$1.794\cdot10^{3}$	$1.98\cdot10^{3}$	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0

FULPKS0Z.MCD 03-10-02

$P := 1, 2 .. 50$

$K := 1, 2 .. 40$

VERTICAL = P
HORIZONTAL = K
TABLE = S

$$B_{P,K} := K + \frac{(P+1)}{2} + \frac{(K^2+K)}{P}$$

$$S_{P,K} := K \cdot (B_{P,K}) + K \cdot \frac{(K+1)}{2}$$

$$U_{P,K} := (\mathrm{floor}(B_{P,K}) - \mathrm{ceil}(B_{P,K}) + 1) \cdot S_{P,K}$$

K

P

U =

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	5	21	54	110	195	315	476
2	0	0	0	0	0	0	0	0
3	0	0	15	33	0	100	153	0
4	0	5	15	0	0	90	135	0
5	0	0	0	0	54	85	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	117	161
8	0	0	0	33	54	0	0	0
9	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0
12	0	0	21	0	0	85	117	0
13	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
15	0	0	0	0	0	90	0	0
16	0	0	0	0	0	0	0	161
17	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0

FULPKBOZ ✓

~~FULCRUMY~~ 03-09-30

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$$B = W(P,K) := K + \frac{(K^2+K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\text{ceil}(W(P,K)) - \text{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > J

B > K + P

K = U

6
P

U =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	~~4~~	~~9~~	~~16~~	~~25~~	~~36~~	~~49~~	~~64~~	~~81~~	~~100~~	~~121~~	~~144~~	~~169~~	~~196~~	~~225~~	~~256~~	~~289~~
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	6	~~9~~	0	~~17~~	~~22~~	0	~~34~~	41	0	~~57~~	~~66~~	0	86	~~97~~	0
4	~~4~~	~~6~~	0	0	15	19	0	0	~~34~~	40	0	0	61	69	0	0
5	0	0	0	11	~~14~~	0	0	0	~~30~~	~~35~~	0	0	0	~~59~~	66	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	16	~~19~~	0	0	0	0	0	~~43~~	~~48~~	0	0
8	0	0	~~9~~	~~11~~	0	0	0	0	0	0	32	36	0	0	0	0
9	0	0	0	0	0	0	0	21	~~24~~	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	26	~~29~~	0	0	0	0	0
12	0	9	0	0	~~14~~	~~16~~	0	0	23	0	0	0	0	38	0	0
13	0	0	0	0	0	0	0	0	0	0	0	31	~~34~~	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	~~15~~	0	0	0	~~23~~	0	0	0	0	36	~~39~~	0
16	0	0	0	0	0	0	~~19~~	~~21~~	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	41
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	~~17~~	0	0	0	~~24~~	~~26~~	0	0	0	35	0	0

square

FULPRKBOZ ✓

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$\beta = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\text{ceil}(W(P,K)) - \text{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	289	324	361	400	441	484	529	576	625	676	729	784	841	900	961	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	121	~~134~~	0	162	~~177~~	0	209	226	0	262	281	0	321	342	0
4	0	96	106	0	0	139	151	0	0	190	204	0	0	249	265	0
5	0	0	0	98	~~107~~	0	0	0	147	~~158~~	0	0	0	206	~~219~~	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	84	~~91~~	0	0	0	0	0	139	~~148~~	0	0	0
8	0	0	0	71	77	0	0	0	0	0	0	126	134	0	0	0
9	0	~~56~~	~~61~~	0	0	0	0	0	0	0	109	~~116~~	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	~~69~~	~~74~~	0	0	0	0	0	0	0	0	0
12	0	49	53	0	0	66	0	0	0	0	91	0	0	108	114	0
13	0	0	0	0	0	0	0	0	0	~~82~~	~~87~~	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	56	0	0	0	72	0	0	0	0	~~95~~	~~100~~	0
16	0	0	0	0	0	0	0	66	70	0	0	0	0	0	0	0
17	41	~~44~~	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	46	~~49~~	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	68	0	0	0	83	87	0

FULPRDOZ ✓

~~FULCRUMY~~ 03-09-30

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$$B \approx W(P,K) := K + \frac{\left(K^2 + K\right)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
1	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	386	409	0	457	482	0	534	561	0	617	646	0	706	737	0
4	0	0	316	334	0	0	391	411	0	0	474	496	0	0	565	589
5	0	0	0	275	290	0	0	0	354	371	0	0	0	443	462	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	208	219	0	0	0	0	0	291	304	0	0	0	0
8	0	0	0	0	197	207	0	0	0	0	0	0	284	296	0	0
9	0	0	0	0	180	189	0	0	0	0	0	0	0	269	280	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	134	141	0	0	0	0	0	0	0	0	0	221	230	0	0
12	0	0	133	0	0	0	0	168	0	0	191	199	0	0	224	0
13	0	0	0	0	0	0	0	159	166	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	127	0	0	0	151	0	0	0	0	184	191	0
16	0	0	0	0	0	0	0	0	145	151	0	0	0	0	0	0
17	0	0	108	~~113~~	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	121	~~126~~	0	0	0	0	0	0	0	0
20	0	0	0	104	0	0	0	0	0	0	0	0	0	0	159	0

FULPKB02 ✓

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
20	0	0	0	0	0	0	0	0	0	68	0	0	0	83	87	0
21	0	0	0	0	51	54	0	0	0	0	0	74	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	56	59	0	0	0	0	0	0	0	0
24	0	0	0	0	50	0	0	0	0	0	0	71	0	0	0	0
25	0	0	0	0	0	0	0	0	61	64	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	66	69	0	0	0	0
28	0	0	0	0	0	52	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	71	74	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	76	79
32	~~41~~	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	~~52~~	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	~~50~~	0	0	0	0	0	0	0	0	0	0	0
36	0	~~44~~	~~46~~	0	0	0	0	0	0	0	64	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	~~64~~	0	0	0	0	0

FULPRBO7

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = \quad W(P,K) := K + \frac{\left(K^2 + K\right)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\mathrm{ceil}(W(P,K)) - \mathrm{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
20	0	0	0	104	0	0	0	0	0	0	0	0	0	0	159	0
21	0	0	0	0	106	0	0	0	0	0	134	139	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	147	152
24	0	0	0	0	100	104	0	0	0	0	0	0	0	139	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	91	0	0	0	0	0	0	117	121	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	79	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	81	84	0	0	0	0	0	0	0	0	0	0	121	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	86	89	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	121	0
37	0	0	0	0	0	91	94	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	96	99	0	0	0	0	0	0	0

FULPRB03

~~FULCRUMY~~ 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$\beta \approx W(P,K) := K + \frac{\left(K^2 + K\right)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := \left(-\left(\text{ceil}(W(P,K)) - \text{floor}(W(P,K)) - 1\right)\right) \cdot W(P,K)$$

B > K + P

K

P

U =

	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
1	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$	$\cdot 10^3$
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	801	834	0	902	937	0	$\cdot 10^3$	$\cdot 10^3$	0	$\cdot 10^3$	$\cdot 10^3$	0	$\cdot 10^3$	$\cdot 10^3$	0
4	589	0	0	664	690	0	0	771	799	0	0	886	916	0	0	$\cdot 10^3$
5	0	0	0	542	563	0	0	0	651	674	0	0	0	770	795	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	388	403	0	0	0	0	0	499	516	0	0	0	0	0
8	0	0	0	0	0	387	401	0	0	0	0	0	0	506	522	0
9	0	0	0	0	0	0	0	376	389	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	330	341	0	0	0	0	0	0
12	0	0	0	0	269	0	0	298	308	0	0	339	0	0	0	0
13	0	0	0	0	0	262	271	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	228	0	0	0	260	0	0	0	0	303	312	0
16	0	0	0	0	0	0	0	0	0	256	264	0	0	0	0	0
17	0	0	0	0	209	216	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	234	241	0	0	0	0
20	0	0	0	182	188	0	0	0	213	0	0	0	0	0	0	0

2

FULCRUMY 03-09-30

VERTICAL = P
HORIZONTAL = K
TABLE = B
THE NUMBERS ARE B'S
THE ZEROS ARE NOT

$P := 1, 2 .. 50$

$K := 1, 2 .. 90$

$$B = W(P,K) := K + \frac{(K^2 + K)}{P} + \frac{(P+1)}{2}$$

$$U_{P,K} := (-(\text{ceil}(W(P,K)) - \text{floor}(W(P,K)) - 1)) \cdot W(P,K)$$

B > K + P

K

P

U =

	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
20	0	0	0	182	188	0	0	0	213	0	0	0	0	0	0	0
21	0	0	171	0	0	0	0	0	0	0	219	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	152	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	174	0	0	0	0	0	0	0	219	225	0
25	0	0	0	160	165	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	173	178	0	0	0	0	0	0	0
28	0	0	0	151	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	186	191	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	199
32	0	134	138	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	161	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	137	0	0	0	0	0	161	0	0	0	0	0	0
36	0	0	0	0	0	0	0	151	155	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	139	0	0	0	0	0	0	0	0	0	0

FULCRUM7.MCD 03-09-25

B := 81

N := 1, 2 .. (B − 1) M := 1

$V(N) := B \cdot N - \frac{(N \cdot (N+1))}{2}$ $S(N) := B \cdot N + \frac{(N \cdot (N+1))}{2}$ $U(N) := B - N$ $H(N) := B + N$

balances
B=81 3 sequence
below above
sum 9 684 = 684 8
65 3120 = 3120 32
77 3234 = 3234 33

L = U+1
L = 2U+1
7L = 3U

$T_{N,M} := V(N)$ $J_{N,M} := U(N)$ $W_{N,M} := S(N)$ $K_{N,M} := H(N)$

J =

	0	1
0	0	0
1	0	80
2	0	79
3	0	78
4	0	77
5	0	76
6	0	75
7	0	74
8	0	73
9	0	72
10	0	71
11	0	70
12	0	69
13	0	68
14	0	67
15	0	66
16	0	65
17	0	64
18	0	63
19	0	62

T =

	0	1
0	0	0
1	0	80
2	0	159
3	0	237
4	0	314
5	0	390
6	0	465
7	0	539
8	0	612
9	0	684
10	0	755
11	0	825
12	0	894
13	0	962
14	0	$1.029 \cdot 10^3$
15	0	$1.095 \cdot 10^3$
16	0	$1.16 \cdot 10^3$
17	0	$1.224 \cdot 10^3$
18	0	$1.287 \cdot 10^3$
19	0	$1.349 \cdot 10^3$

W =

	0	1
0	0	0
1	0	82
2	0	165
3	0	249
4	0	334
5	0	420
6	0	507
7	0	595
8	0	684
9	0	774
10	0	865
11	0	957
12	0	$1.05 \cdot 10^3$
13	0	$1.144 \cdot 10^3$
14	0	$1.239 \cdot 10^3$
15	0	$1.335 \cdot 10^3$
16	0	$1.432 \cdot 10^3$
17	0	$1.53 \cdot 10^3$
18	0	$1.629 \cdot 10^3$
19	0	$1.729 \cdot 10^3$

K =

	0	1
0	0	0
1	0	82
2	0	83
3	0	84
4	0	85
5	0	86
6	0	87
7	0	88
8	0	89
9	0	90
10	0	91
11	0	92
12	0	93
13	0	94
14	0	95
15	0	96
16	0	97
17	0	98
18	0	99
19	0	100

FULCRUM7.MCD 03-09-25

$B := 81$

$N := 1, 2 .. (B - 1)$ $M := 1$

$V(N) := B \cdot N - \frac{(N \cdot (N + 1))}{2}$ $S(N) := B \cdot N + \frac{(N \cdot (N + 1))}{2}$ $U(N) := B - N$ $H(N) := B + N$

$T_{N,M} := V(N)$ $J_{N,M} := U(N)$ $W_{N,M} := S(N)$ $K_{N,M} := H(N)$

J =

	0	1
51	0	30
52	0	29
53	0	28
54	0	27
55	0	26
56	0	25
57	0	24
58	0	23
59	0	22
60	0	21
61	0	20
62	0	19
63	0	18
64	0	17
65	0	16
66	0	15
67	0	14
68	0	13
69	0	12
70	0	11

T =

	0	1
61	0	$3.05 \cdot 10^{3}$
62	0	$3.069 \cdot 10^{3}$
63	0	$3.087 \cdot 10^{3}$
64	0	$3.104 \cdot 10^{3}$
65	0	$3.12 \cdot 10^{3}$ +
66	0	$3.135 \cdot 10^{3}$
67	0	$3.149 \cdot 10^{3}$
68	0	$3.162 \cdot 10^{3}$
69	0	$3.174 \cdot 10^{3}$
70	0	$3.185 \cdot 10^{3}$
71	0	$3.195 \cdot 10^{3}$
72	0	$3.204 \cdot 10^{3}$
73	0	$3.212 \cdot 10^{3}$
74	0	$3.219 \cdot 10^{3}$
75	0	$3.225 \cdot 10^{3}$
76	0	$3.23 \cdot 10^{3}$
77	0	$3.234 \cdot 10^{3}$ ▲
78	0	$3.237 \cdot 10^{3}$
79	0	$3.239 \cdot 10^{3}$
80	0	$3.24 \cdot 10^{3}$

W =

	0	1
18	0	$1.629 \cdot 10^{3}$
19	0	$1.729 \cdot 10^{3}$
20	0	$1.83 \cdot 10^{3}$
21	0	$1.932 \cdot 10^{3}$
22	0	$2.035 \cdot 10^{3}$
23	0	$2.139 \cdot 10^{3}$
24	0	$2.244 \cdot 10^{3}$
25	0	$2.35 \cdot 10^{3}$
26	0	$2.457 \cdot 10^{3}$
27	0	$2.565 \cdot 10^{3}$
28	0	$2.674 \cdot 10^{3}$
29	0	$2.784 \cdot 10^{3}$
30	0	$2.895 \cdot 10^{3}$
31	0	$3.007 \cdot 10^{3}$
+ 32	0	$3.12 \cdot 10^{3}$
▲ 33	0	$3.234 \cdot 10^{3}$
34	0	$3.349 \cdot 10^{3}$
35	0	$3.465 \cdot 10^{3}$
36	0	$3.582 \cdot 10^{3}$
37	0	$3.7 \cdot 10^{3}$

K =

	0	1
0	0	0
1	0	82
2	0	83
3	0	84
4	0	85
5	0	86
6	0	87
7	0	88
8	0	89
9	0	90
10	0	91
11	0	92
12	0	93
13	0	94
14	0	95
15	0	96
16	0	97
17	0	98
18	0	99
19	0	100

fultest FULCRUM NUMBERS JUNE 29, 2006

$B := 61$ $n := 1, 2 .. 40$ $m := 1, 2 .. 80$

$U(n) := n \cdot B + n \cdot \frac{(n+1)}{2}$ $L(m) := m \cdot B - m \cdot \frac{(m+1)}{2}$

$W_{n,m} := U(n) - L(m)$

L

U

W =

	16	17	18	19	20	21	22	23	24	25	26	27	28
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	-778	-822	-865	-907	-948	-988	-1027	-1065	-1102	-1138	-1173	-1207	-1240
2	-715	-759	-802	-844	-885	-925	-964	-1002	-1039	-1075	-1110	-1144	-1177
3	-651	-695	-738	-780	-821	-861	-900	-938	-975	-1011	-1046	-1080	-1113
4	-586	-630	-673	-715	-756	-796	-835	-873	-910	-946	-981	-1015	-1048
5	-520	-564	-607	-649	-690	-730	-769	-807	-844	-880	-915	-949	-982
6	-453	-497	-540	-582	-623	-663	-702	-740	-777	-813	-848	-882	-915
7	-385	-429	-472	-514	-555	-595	-634	-672	-709	-745	-780	-814	-847
8	-316	-360	-403	-445	-486	-526	-565	-603	-640	-676	-711	-745	-778
9	-246	-290	-333	-375	-416	-456	-495	-533	-570	-606	-641	-675	-708
10	-175	-219	-262	-304	-345	-385	-424	-462	-499	-535	-570	-604	-637
11	-103	-147	-190	-232	-273	-313	-352	-390	-427	-463	-498	-532	-565
12	-30	-74	-117	-159	-200	-240	-279	-317	-354	-390	-425	-459	-492
13	44	0	-43	-85	-126	-166	-205	-243	-280	-316	-351	-385	-418
14	119	75	32	-10	-51	-91	-130	-168	-205	-241	-276	-310	-343
15	195	151	108	66	25	-15	-54	-92	-129	-165	-200	-234	-267
16	272	228	185	143	102	62	23	-15	-52	-88	-123	-157	-190
17	350	306	263	221	180	140	101	63	26	-10	-45	-79	-112
18	429	385	342	300	259	219	180	142	105	69	34	0	-33
19	509	465	422	380	339	299	260	222	185	149	114	80	47

$B := 69$ $k := 27$

$Y(k) := k \cdot B + k \cdot \frac{(k+1)}{2}$ $Y(k) = 2241$

61	L	U
	17	13
	27	18
	49	24

fi 1test
NAME?

FULCRUM NUMBERS JUNE 29, 2006

$B := 16$

$n := 1, 2 .. 20$ $m := 1, 2 .. 20$

$U(n) := n \cdot B + n \cdot \frac{(n+1)}{2}$ $L(m) := m \cdot B - m \cdot \frac{(m+1)}{2}$

$W_{n,m} := U(n) - L(m)$

L m

U

n

W =

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	-12	-25	-37	-48	-58	-67	-75	-82	-88	-93	-97	-100	-102	-103
2	0	20	6	-7	-19	-30	-40	-49	-57	-64	-70	-75	-79	-82	-84	-85
3	0	39	25	12	0	-11	-21	-30	-38	-45	-51	-56	-60	-63	-65	-66
4	0	59	45	32	20	9	-1	-10	-18	-25	-31	-36	-40	-43	-45	-46
5	0	80	66	53	41	30	20	11	3	-4	-10	-15	-19	-22	-24	-25
6	0	102	88	75	63	52	42	33	25	18	12	7	3	0	-2	-3
7	0	125	111	98	86	75	65	56	48	41	35	30	26	23	21	20
8	0	149	135	122	110	99	89	80	72	65	59	54	50	47	45	44
9	0	174	160	147	135	124	114	105	97	90	84	79	75	72	70	69
10	0	200	186	173	161	150	140	131	123	116	110	105	101	98	96	95
11	0	227	213	200	188	177	167	158	150	143	137	132	128	125	123	122
12	0	255	241	228	216	205	195	186	178	171	165	160	156	153	151	150
13	0	284	270	257	245	234	224	215	207	200	194	189	185	182	180	179
14	0	314	300	287	275	264	254	245	237	230	224	219	215	212	210	209
15	0	345	331	318	306	295	285	276	268	261	255	250	246	243	241	240
16	0	377	363	350	338	327	317	308	300	293	287	282	278	275	273	272
17	0	410	396	383	371	360	350	341	333	326	320	315	311	308	306	305
18	0	444	430	417	405	394	384	375	367	360	354	349	345	342	340	339
19	0	479	465	452	440	429	419	410	402	395	389	384	380	377	375	374

$L(4) = U(3)$ 54
$L(13) = U(6)$ 117

FULCRUM TEST 2

$B := 16$

$n := 1, 2 .. 20$

$U(n) := n \cdot B + n \cdot \frac{(n+1)}{2}$

$m := 1, 2 .. 20$

$L(m) := m \cdot B - m \cdot \frac{(m+1)}{2}$

n =	U(n) =	U(n + 20) =
1	17	567
2	35	605
3	54 ✓	644
4	74	684
5	95	725
6	117 ✓	767
7	140	810
8	164	854
9	189	899
10	215	945
11	242	992
12	270	1040
13	299	1089
14	329	1139
15	360	1190
16	392	1242
17	425	1295
18	459	1349
19	494	1404
20	530	1460

L(m) =	L(m + 20) =	L(m + 40) =	L(m + 60) =
15	105	-205	-915
29	99	-231	-961
42	92	-258	-1008
54 ✓	84	-286	-1056
65	75	-315	-1105
75	65	-345	-1155
84	54	-376	-1206
92	42	-408	-1258
99	29	-441	-1311
105	15	-475	-1365
110	0 ✓	-510	-1420
114	-16	-546	-1476
117 ✓	-33	-583	-1533
119	-51	-621	-1591
120	-70	-660	-1650
120	-90	-700	-1710
119	-111	-741	-1771
117 ✓	-133	-783	-1833
114	-156	-826	-1896
110	-180	-870	-1960

$B(s,n) = B_1(s) + \Delta B(s) \cdot (n-1)$ $B_1(s) = s^2$ $\Delta B(s) = s^2 + (s+1)^2$

$= s^2 + (n-1)[s^2 + (s+1)^2]$ $= s^2 + (n-1)[2s^2 + 2s + 1]$

$= s^2 + ns^2 - s^2 + n(s^2+1) - (s^2+1)$

$= ns^2 + (n-1)(s^2+1)$ ✓

$QB = \Delta B \cdot P \pm (2s+1)$

OLD

$\Delta B \cdot (2n-1) \pm (2s+1)$

$(2s^2 + 2s + 1)(2\underset{p}{n} - 1) \pm (2s+1)$

MAVERICK Bs

15
~~17~~
23
~~30~~
32
38
40
~~43~~
50
52
53

$P = 1$ Set B_1's at $P = 1$

s	B_1	ΔB	$\underset{a}{K} = aP$	$\underset{b}{J} = bP$
1	1	5		
2	4	5 13	1	2
3	9	13 25	2	3
4	16	25 41	3	4
5	25	41	4	5

$P = 2n - 1$ 1, 3, 5
$n = 1 \;\; 2 \;\; 3$

2 ΔB's for each s

$J = P + K$ $B_1(s) = s^2$

$$\Delta B = s^2 + (s \pm 1)^2 \begin{cases} 2s^2 + 2s + 1 \\ 2s^2 - 2s + 1 \end{cases}$$

$s = 1$

	$B(s)J$	K	P	ΔB
$n = 1$ $B_1 =$	1		1	5
$n = 2$	6		3	"
$n = 3$	11		5	"

s	$\overset{+}{\Delta B}$	$\overset{-}{\Delta B}$
1	5	1
2	13	5
3	25	13
4	41	25

$\Delta B = 2s^2 \pm 2s + 1$
$s = 1, 2, ---$

Those that originate B_1's at $P = 3$

FULCRUM SEQUENCES [of recursion type $B_n = B_{n-1} + \text{Const}$

Initial P = 1 FAMILY = SQUARE DERIVED

INITIAL VALUES

$J = K + P$

		B	J	K	P	ΔB	ΔJ	ΔK	ΔP	
	Init	1	0	0	1	5	4	2	2	
II	2°	6	5	2	3					
	3°	11	9	4	5	1, 6, 11,	16, 21, 26,	31, ...		
	Init	4	2	1	1	5	4	2	3	
I		9	6	3	3					
		14	10	5	5	4, 9, 14,	19, 24, 29	34		
	Init	4	2	1	1	13	6	4	2	
II		17	8	5	3					
		30	14	9	5	4, 17, 30	43, 56, 69	82		
	Init	9	3	2	1	13	6	4	2	
I		22	9	6	3					
		35	21	10	5	9, 22, 35	48, 61, 74	87		
	Init	9	3	2	1	25	8	6	2	
II		34	11	8	3					
		59	19	14	5	9, 34, 59	84, 109, 134	159		
	Init	16	4	3	1	25	8	6	2	
I		41	12	9	3					
		66	20	15	5	16, 41, 66	91, 116, 141	166		
	Init	16	4	3	1	41	10	8	2	
II		57	14	11	3					
		98	24	19	5	16, 57, 98	139, 180, 221	262		
	Init	25	5	4	1	41	10	8	2	
I		66	15	12	3					
		107	25	20	5	25, 66, 107	148, 189, 230	, 271		
	Init	25	5	4	1	61	12	10	2	
II		86	17	14	3					
		147	29	24	5	25, 86, 147	, 208, 269	330, 391		

FORMULA I $B(s,n) = ns^2 + (n-1)(s-1)^2$

FORMULA II $B(s,n) = ns^2 + (n-1)(s+1)^2$

In addition ∃ the □ sequence itself 1, 4, 9, 16, ...

Δ13	Δ13	Δ5	Δ5
④ B	⑨ B	③ B	⑤ B
9	4	4	6
☒ 22	17	☒ 9	11
35	30	14	16
48	43	19	21
61	56	24	26
74	69	29	31
87	82	34	36
100	95	39	41
113	108	44	46
126	121	49	51
139	134	54	56
152	147	59	61
		64	66

RENAME SEQUENCES
① = J = B-1
② = SQUARES
③ = 5:4 X
④ = 5:6 +
⑤ = 13:9 X
⑥ = 13:4 +
⑦ = 17:19 X
⑧ = 17:15 +
⑨ = 25:16 X
⑩ = 25:9 +
⑪ = 29:23

Δ17	Δ17	Δ25	Δ25	Δ29
⑥ B	⑩ B	⑦ B	⑪ B	⑧ B
19	15	16	9	23
☒ 36	32	☒ 41	34	52
53	49	66	59	81
70	66	91	84	110
87	83	116	109	139
104	100	141	134	168
121	117	166	159	197
138	134	191	184	226
155	151	216	209	255
172	168			284
189	185			313
206	202			342

FULCRUMP.MCD 03-09-28

$B := 4, 5 .. 122$

$P < B - K$

$K := 1, 2 .. 60$

$$P_{B,K} := \frac{1}{2}\cdot\left[(2\,(B-K)-1)-\sqrt{(2\,(B-K)-1)^2-8\,\left(K^2+K\right)}\right]$$

K

B

P =

	4	5	6	7	8	9	10	11	12	13	14	15	16	17
21	1.26	2.07	3.26	5.12	9	-6.91i	0.48i	3.18i	5.48i	7.54i	9.44i	1.21i	2.89i	4.49i
22	1.18	1.93	3	4.59	7.32	-4.87i	-9.37i	-12.4i	4.89i	7.08i	9.07i	0.92i	2.67i	4.33i
23	1.11	1.81	2.78	4.18	6.36	12	-7.98i	1.48i	-14.2i	6.55i	8.65i	0.59i	-22.4i	4.12i
24	1.05	1.7	2.59	3.84	5.69	9	-6.14i	0.38i	3.41i	5.93i	8.16i	0.19i	2.09i	3.87i
25	1	1.6	2.43	3.56	5.18	7.74	-3.12i	-9.04i	2.48i	5.22i	-17.6i	9.74i	1.72i	3.57i
26	0.95	1.52	2.29	3.33	4.76	6.9	11	-7.33i	1.39i	4.41i	6.96i	9.23i	-21.3i	3.23i
27	0.91	1.44	2.16	3.12	4.42	6.26	9.27	-4.87i	0.09i	3.48i	6.24i	8.65i	0.83i	2.84i
28	0.87	1.38	2.05	2.94	4.13	5.76	8.21	13.63	-8.47i	-12.4i	5.42i	7.99i	0.29i	-22.4i
29	0.83	1.31	1.95	2.79	3.88	5.35	7.44	11	5-6.3i	1.12i	4.48i	7.26i	9.69i	-21.9i
30	0.8	1.26	1.86	2.64	3.66	5	6.84	9.65	5-2.4i	-9.58i	3.41i	6.42i	9.02i	1.35i
31	0.77	1.2	1.78	2.52	3.47	4.7	6.35	8.72	13	5-7.6i	2.16i	5.48i	8.27i	0.73i
32	0.74	1.16	1.7	2.4	3.29	4.44	5.94	8	11.24	-4.66i	0.67i	4.41i	7.43i	0.04i
33	0.71	1.11	1.64	2.3	3.14	4.21	5.58	7.42	10.1	15.47	-8.82i	3.18i	6.48i	9.28i
34	0.69	1.07	1.57	2.2	3	4	5.27	6.94	9.24	13	5-6.3i	1.74i	5.42i	8.43i
35	0.66	1.04	1.51	2.12	2.87	3.81	5	6.52	8.56	11.59	20	-9.99i	-14.2i	7.49i
36	0.64	1	1.46	2.04	2.76	3.65	4.76	6.16	8	10.57	15	-7.73i	-12.8i	6.42i
37	0.62	0.97	1.41	1.96	2.65	3.49	4.54	5.85	7.52	9.78	13.21	-4.21i	1.12i	5.22i
38	0.6	0.94	1.36	1.89	2.55	3.36	4.34	5.57	7.11	9.13	12	17.38	-9.04i	3.85i
39	0.58	0.91	1.32	1.83	2.46	3.23	4.16	5.31	6.75	8.58	11.07	15	-6.14i	2.24i
40	0.57	0.88	1.28	1.77	2.38	3.11	4	5.09	6.42	8.11	10.33	13.53	20.63	0.28i

Remaining. ON FIRST PAGE

B J K P	Δs	Sequen
15 9 5 4	17 10 6 4	19, 32, 49, 66, 83
19 10 6 4	17 10 6 4	19, 36, 53, 70, 87
34 13 9 4		
40 14 10 4		
61 17 13 4		
69 18 14 4		

32 19 11 8
34 ~~13~~ 9 4

35 34 14 20

36 20 12 8
49 29 17 13
50 44 20 24
52 49 21 28
53 30 18 12

23 21 9 12 $\Delta B = 29$ 23, 52, 81, 110, 139
38 26 14 12

35 34 14 20

Page 2   56 35 20 15

FULCRUM SEQUENCES

All subsumed by □'s

	B	J	K	P	ΔB	ΔJ	ΔK	ΔP	
Init	6	5	2	3	5	4	2	2	
	11		SAME	AS	P=1	1101	5422		
	16			Belongs	to	1101	5422		
Init	9	6	3	3					
	Belongs	to	9321	13642					
Init	17	8	5	3					
	"	Belongs	to	4211	13642				
	"								
Init	22	9	6	3					
	"	Belongs	to	9,32,1	13,6,4,2				
	"								
Init	11	9	4	5					
	"	Belongs	to	1101	5422				
	"								
Init	14	10	5	5					
	"	Belongs	to	4211	5422				
	"								
Init	16	13	6	7					
	"	Belongs	to	1101	5422				
	"								
Init	19	14	7	7					
	"	Belongs	to	4211	5422				
	"								
Init	21	17	8	9					
		Belongs	to	1101	5422				

SEQUENCES

① $J = B - 1$

---

② $\boxed{2\ 1\ 1}$

#	B	J	K	P	Σ
1	4	2	1	1	5
N	$(N+1)^2$	N+1	N	1	

$\Delta^3 \Sigma = 6$

B LAST NUMBERS

1, 4 9 6 5 6 9 4 1 0 1…

$\boxed{10}$

---

③ $\boxed{2\ 1\ 1}$

#	B	J	K	P	Σ
1	4	2	1	1	5
N	$4+5(N-1)$	$2(2N-1)$	$2N-1$	$2N-1$	

$\Delta^2 \Sigma = 24$

4, 9, 4, 9, 4, 9…

$\boxed{2}$

---

④ $\boxed{3\ 2\ 1}$

#	B	J	K	P	Σ
1	9	3	2	1	21
N	$9+(N-1)\cdot 13$	$3(2N-1)$	$2(2N-1)$	$2N-1$	

$\Delta^2 \Sigma = 120$

9, 2, 5, 8, 1, 4, 7, 0, 3, 6, 9

$\boxed{10}$

---

⑤ $\boxed{5\ 2\ 3}$

#	B	J	K	P	Σ
1	6	5	2	3	15
N	$6+5(N-1)$	$4N+1$	$2N$	$2N+1$	

$\Delta^2 \Sigma = 24$

6, 1, 6, 1, 6, 1, …

$\boxed{2}$

---

⑥ $\boxed{5\ 3\ 2}$

#	B	J	K	P	Σ
1	19	10	6	4	135
N	$19+17\cdot(N-1)$	$10N$	$6N$	$4N$	

$\Delta^2 \Sigma = 240$

9, 6, 3, 0, 7, 4, 1, 8, 5, 2, 9

$\boxed{10}$

---

⑦ $\boxed{4\ 3\ 1}$

#	B	J	K	P	Σ
1	16	4	3	1	54
N	$16+25\cdot(N+1)$	$4(2N-1)$	$3(2N-1)$	$2N-1$	

$\Delta^2 \Sigma = 336$

6, 1, 6, 1 …

$\boxed{2}$

---

⑧ $\boxed{7\ 3\ 4}$

#	B	J	K	P	Σ
1	23	21	9	12	252
N	$23+29N$	$7(4N-1)$	$3(4N-1)$	$4(4N-1)$	

$\Delta^2 \Sigma = 840$

3, 2, 1, 0, 9, 8, 7, 6, 5, 4, 3,

$\boxed{10}$

---

⑨ $\boxed{2\ 1\ 1}$

FULCRUMP.MCD 03-09-28

$B := 4, 5 .. 120$

$P < B - K$

$K := 1, 2 .. 60$

$$P_{B,K} := \frac{1}{2}\cdot\left[(2\,(B-K)-1)-\sqrt{(2\,(B-K)-1)^2-8\,\left(K^2+K\right)}\right]$$

K (= U)

B

P =

	2	3	4	5	6
6	3	2.5-4.21i	1.5-6.14i	0.5-7.73i	-0.5-9.15i
7	1.63	3.5-3.43i	2.5-5.81i	1.5-7.6i	0.5-9.15i
8	1.23	4.5-1.94i	3.5-5.27i	2.5-7.33i	1.5-9.04i
9	1	3	4.5-4.44i	3.5-6.91i	2.5-8.82i
10	0.85	2.23	5.5-3.12i	4.5-6.3i	3.5-8.47i
11	0.74	1.82	5	5.5-5.45i	4.5-7.98i
12	0.65	1.55	3.47	6.5-4.21i	5.5-7.33i
13	0.59	1.36	2.82	7.5-1.94i	6.5-6.46i
14	0.53	1.21	2.41	5	7.5-5.27i
15	0.49	1.1	2.12	4	8.5-3.43i
16	0.45	1	1.9	3.41	7
17	0.42	0.92	1.72	3	5.38
18	0.39	0.85	1.57	2.69	4.55
19	0.37	0.79	1.45	2.44	4
20	0.35	0.74	1.35	2.24	3.59
21	0.33	0.7	1.26	2.07	3.26
22	0.31	0.66	1.18	1.93	3
23	0.29	0.63	1.11	1.81	2.78
24	0.28	0.59	1.05	1.7	2.59
25	0.27	0.57	1	1.6	2.43

Fulcrums

∃ more inverses than we realize
Symmetries
Watersheds

Lewis Carroll's
Through the Looking Glass

Negative Numbers $x \longleftrightarrow -x$

rational $x \longleftrightarrow \frac{1}{x}$ $x^n \longleftrightarrow x^{-n}$

irrational $x^n \longleftrightarrow \sqrt[n]{x}$ or $x^n \longleftrightarrow x^{1/n}$

~~$\log_a x \longleftrightarrow a$~~ $x^n \longleftrightarrow \log_x n$

Emmy Noether
Conservation Principles

Zwicky

The Planck particle as fulcrum

Airplanes WW II
bullet holes

The Pascal Triangle [YANG HUI]
and its addition to subtraction inverse
→ Fractal

Play Pen

Prime △ Triangle ~ some of Wolfram's semi-random

04-01-14

FULCRUM NUMBERS.

Type 1 Fulcrum is a number — Symbols B

Type 2 Fulcrum lies between two number — W

Watershed Number

Example — W

1° number = number of terms on right squared

$4+5+6 = 7+8$ [=15]

$9+10+11+12 = 13+14+15$ [42]

$16+17+18+19+20 = 21+22+23+24$

B

$6+7+8$ ⑨ $10+11$ [21]

$2+3$ ④ 5 [5]

A special Case: the first number must be 1

$1+2 = 3$ [3]

What is the next ~~number~~ sequence of this description?

$1+2+\cdots+n = (n+1)+\cdots x$?

Generalized! W

$1+2+3+4+5$ ⑥ $7+8$ [15]

THESE B NUMBERS WILL BE CALLED RAMANUJAN NUMBERS!

$B_0 = 0,\ B_1 = 1,\ B_2 = 6$

$B_n = 6B_{n-1} - B_{n-2}$

RECURSION FORMULA

EXPLICIT FORMULA

$$B_n = \frac{1}{2\sqrt{8}}\left[p^n - q^n\right]$$

where $p = 3+\sqrt{8}$
$q = 3-\sqrt{8}$

See Dr. Matrix p28 & 242

The above are linear

Fulcrums may also be of squares

First number $= n(2n+1)$ where is n is the number of terms on the right

$3^2 + 4^2 = 5^3$ ⊛ [25]

$10^2 + 11^2 + 12^2 = 13^2 + 14^2$ [365]

$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$

$36^2 + 37^2 + \cdots = \cdots + 44^2$

$55^2 + 56^2 + \cdots = \cdots + 65^2$

cubes? $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$

⊛ Pythagoras + Fermat's last theorem

UNSEQUENCED

#	B	Σ	J	K	P	SEQUENCE		
	15	90	9	5	4	J = 2P+1	K = P+1	9 5 4
	~~16~~	~~117~~	~~13~~	~~6~~	~~7~~	~~5~~		
⑨	17	100	8	5	3			8 5 3
⑨	30	315	14	9	5			14 9 5
⑩	32	418	19	11	8			19 11 8
also ③ 26-13-13	⑪ 34	308	11	8	3	J = 3P+2	K = 2P+2	11 8 3
	34 ~~36~~	351 ~~609~~	13 ~~29~~	9 ~~14~~	4 ~~13~~	~~5~~		13 9 4
	38	637	26	14	12			13 7 6
	40	455	14	10	4	J+K/6		7 5 2
⑨	43	650	20	13	7			20 13 7
⑩	49	986	29	17	12			29 17 12
also ~~[illegible]~~ 9 ⑤'s	56	1330	35	20	15	J = 2P+5	K = P+5	7 4 3
stet ⑨	~~56~~	~~1105~~	~~26~~	~~17~~	~~9~~	J = 3P-1	K = 2P-1	stet 26 17 9
	57	693	14	11	3			14 11 3
also ⑤, ④	61	884	17	13	3	J+K/10		17 13 3
also ③ ②	64	2015	62	26	36			31 13 18
also ⑤	66	1617	33	21	12	J = 3P-3	K = 2P-3	11 7 4
and ⑦	66	870	15	12	3	J+K/9		5 4 1
	68	2025	45	25	20			9 5 4
also	69	1680	32	21	11			32 21 11
③	69	1071	18	14	4	J+K/8		9 7 2
~~stet~~ dup also ⑤	~~56~~	~~1105~~	~~26~~	~~17~~	~~9~~	~~J = 3P-1~~	~~K = 2P-1~~	~~26 17 9~~

2 71's
72
87
2 91's

4's 91
66

3's 36
49
56
61
64
69
71

81
87

DISCRETE INTEGRATION
[SUMMATION]

KEEP

$$\text{IF } \Delta^3 \Sigma = A$$

$$\Delta^2 \Sigma = C_1 + NA$$

$$\Delta \Sigma = C_2 + NC_1 + \frac{N(N-1)}{2!} A$$

$$\Sigma = C_3 + NC_2 + \frac{N(N-1)}{2!} C_1 + \frac{N(N-1)(N-2)}{3!} A$$

In general $\Sigma = K_0 + \frac{N!}{(N-1)!} K_1 + \frac{N!}{(N-2)!2!} K_2 + \frac{N!}{(N-3)!3!} K_3 + \cdots$

$\delta = 3$ Fulcrums

$24^2 + 7^2 = 25^2$

These B's are related to the Pythagorean Triples

$22 - 6 = \overset{4^2}{16}$, $34 - 9 = \overset{5^2}{25}$, ~~41 - 17 = 24~~ $66 - 17 = \overset{7^2}{49}$ $66 - 57 = \overset{3^2}{9}$ $66 - 41 = \overset{5^2}{25}$

57
15
39

0
0

B			L	U	V				
6 1	3	5	5 1	2 1	15	18			11 ∆
9 2	8 11		6 2	3 2	[33]	67 85	-32		
17 3	5		8 1	5 1	100	53	+102	+ 134	70
22 4	12		9 2	6 2	153	155	-49	151	53
34 5	17		11 1	8 1	308	106	+173	222	124
41 6	16		12 2	9 2	414	279	-102	275	71
57 7	9		14 1	11 1	693	177			
66 8	B+V	V-B	15	12	870			$\delta = 3$	B's
	21	9 13				-	4,8	$B_4 - B_1$	$= 4^2$
+54 21	42	24 59	46			+	6·17	$B_8 - B_7$	$= 3^3$
-23 75	117	83 48	-11			-	7,7	$B_5 - B_2$	$= 5^2$
+109 68	175	131 143	+95	-55	+220	+	173	$B_8 - B_6$	$= 5^2$
-54 167	342	274 99	-44	×4 -220		-	6,17	$B_8 - B_3$	$= 7^3$
+182 113	455	373 263	+174					$B_7 - B_6$	$= 4^2$
-109 295	750	636 168	-95	0	3 4 5				
186	936	804		144 49	12				
				95					

17
71
53
f

~~109~~ 23
54
-97

182
23
205

182
23
159
53
205
41

9

23
8
184

174 -55 = 11 + 44
46 4
220 -220

Subtle Symmetries

$B_8 - B_6 = 5^2$

$-B_8 + B_7 = -3^2$

$B_7 - B_8 = 5^2 - 3^2 = 4^2$

B_6

Squares $\delta = 1$

B L U V
4 5
9 7 2 1 5 16
16 3 2 21 33 17 6
9 4 3 33 23
25 64 56 29 4 ∴ $an^3 + bn^2 + c^2 + d = V$
11 5 4 110 85 6
36 13 195 510 35 6
49 15 120 41
64 17 315 161
81 476 1925
19
100 21
121

$12n^2 + 5n + 4 = V + 13n + 3$

$12n^2 + B = V + 13n + 3$

$12n^2 - 13n - 3 = V - B$

$V + 5n - 1 - 13n - 3 = V - B$

$+ 5n + 4 = B$

$12n^2 - V = 13n + 3 - B$

$V - 12n^2 = B - 13n - 3$

The $\frac{L}{U} = 2$ Fulcrum series

2

B	δ	L	δ	U	δ	V	δ	δ
4	5	2	4	1	2	5	28	
9		6		3		33	52	24
14		10		5		85	76	24
19		14		7		161	100	24
24		18		9		261	124	24
29		22		11		385	148	24
34		26		13		533	172	24
39		30		15		705	196	24
44		34		17		901	220	24
49		38		19		1121	244	24
54		42		21		1365	268	24
59		46		23		1633	292	24
64		50		25		1925	316	24
69		54		27		2241		

$B = 4 + n5$ $L = 2 + n \cdot 4$ $U = 1 + n \cdot 2$

$\Rightarrow an^2 + bn + c = V$

$a + b + c = 5$

$4a + 2b + c = 33$

$9a + 3b + c = 85$

$$\boxed{12n^2 - 8n + 1 = V}$$

$B = 4 + 5n$

9   3 −2

34 also has 11−8, 13−9, 26−13

49 " " 29−17, 38−19, 7−6

---

next $\frac{3}{2}$ B = 9, 22, 35,

FULCRUM SQUARES

n	B	V
1	4	5
2	9	21
3	16	54
4	25	110
5	36	195

$B = (n+1)^2$, $2V = 2n^3 + 5n^2 + 3n$

$2Z = 2n^3 - n^2 - n$

$LW = \sqrt{B}$, $V = \sqrt{B}$ [illegible] 1

FULCRUMS vs. PRIMES

ΔF	~~#~~	Fulcrum	Not Fulcrum	Δ Not	PRIME	NOT PRIME ~~Fulcrum~~	Δ not ~~Not Fulcr~~	2	1
2	1	4	1	1	1 2	4	2	3	
3	2	6	2	1	2 3	6	2	8	
2	3	9	3	•• 2	3 5	8	3		
3	4	11	5	2	5 7	9	1		3
1	5	14	7	1	7 11	10	1		
1	6	15	8	2	11 13	12	2	2	
1	7	16	10	•• 2	13 17	14	2		
2	8	17	12	1	17 19	15	1		2
2	9	19	13	•• 5	19	16	1		
1	10	21	18	2	23	18	2	2	
1	11	22	20	7	29	20	2		
1	12	23	27	1	31	21	1		2
1	13	24	28 33	5	37	22	1		
1	14	25	37	4	41	24	2	1	
3	15	26	42	5	43	25	1		4
1	16	29	45	3	47	26	1		
1	17	30	~~46~~	2	53	27	1		
1	18	31	47	8	59	28	1		
2	19	32	[illegible]	·	61	30	2	2	
1	20	34	55	•• 3	67	32	2		
1	21	35	58	•• 2	71	33	1		4
2	22	36	60	••• 2	73	34	1		
1	23	38	62	1	79	35	1		
1	24	39	63	2	83	36	1		
1	25	40	65	2	89	38	2	1	
2	26	41	67	••	97	39	1		2
1	27	43			101	40	1		
2	28	44 *46			103	42	3	2	
2	29	48			109	44	3		
1	30	49 50		•••	113	45	1		2
1	31	51			127	46	1		
1	32	52			131	48	2	1	
1	33	53			137	49	1		4
1	34	54			139	50	1		
1	35	56		•••	149	51	1		
2	36	57			151	52	1		
1	37	59			157	54	2	1	
2	38	61		•••	163	55	1		4
2	39	64		•••	167	56	1		
3	40	66		••••	173	57	1		
2	41				179	58	1		
	42				181	60	2	2	
	43				191	62	2		
	44				193	63	1		4
	45				197	64	1		
	46				199	65	1		
	47				211	66	1		
	48				223	68	2	1	
	49				227	69	1		

#	PRIME	VULCRUM	NOT FULCRUM	NOT PRIME	MATCH		#	PRIME	FULCRUM	NOT FULCRUM	NOT PRIME	MATCH F-P̄
					P - F	F-P̄						
1	2	4	2	4	11	4	18	61	31	46	28	40
2	3	6	3	6	17	6	19	67	32	47	30	48
3	5	9	5	8	19	9	20	71	34	50	32	49
4	7	11	7	9	23	14	21	73	35		33	
5	11	14	8	10	29	15	22	79	36		34	
6	13	15	10	12	31	16	23	83	38		35	
7	17	16	12	14	41	21	24	89	40		36	
8	19	17	18	15	43	22	25	97	41		38	
9	23	19	20	16	53	24	26	101	43		39	
10	29	21	27	18		25	27	103	48		40	
11	31	22	28	20		26	28	107	49		42	
12	37	23	33	21		30	29	109	51		44	
13	41	24	37	22		32	30	113	52		45	
14	43	25	39	24		34	31	127	53		46	
15	47	26	42	25		35	32	131	54		48	
16	53	29	44	26		36	33	137			~~40~~ 49	
17	59	30	45	27		38	34	139			50	

$\pi(x)$ = the number of primes $\leq x$

$\pi(100) = 25$
200 = 46
300 = 62
400 = 78
500 = 96
600 = 110
700 = 125
800 = 140
900 = 155
1000 = 168

$\exists$ 23 B's < 60
17 P's < 60
6 same

A DIFFERENT FULCRUM!
"FULCRUM NUMBERS"
WITH SYMMETRY

[= all + integers]

N = any + integer; $a = 1, 2, 3, \ldots$; $\check{\sum} a$ = sum of a numbers next $< N$

$\hat{\sum} a$ = sum of a number next $> N$

a	
1	$\check{\sum} a + \hat{\sum} a = 2N$
2	$\check{\sum} 2a + \hat{\sum} 2a = 4N$
3	$\check{\sum} 3 + \hat{\sum} 3 = 6N$
a	$\check{\sum} a + \hat{\sum} a = 2aN$

For all N

~~$N - \check{\sum} = \hat{\sum} - N$~~ B

a		B	
1	$\hat{\sum} 1 - \check{\sum} 1 =$	2	= a
2	$\hat{\sum} 2 - \check{\sum} 2 =$	6	3a
3	$\hat{\sum} 3 - \check{\sum} 3 =$	12	4a
4	$\hat{\sum} 4 - \check{\sum} 4 =$	20	5a

$$B = \frac{a(a+1)}{2}$$

7-15-2012

RAMANUJAN
NUMBERS

WHEN $J = B - 1$

(or $L = 1$) (Count begins at 1)

THIS SPECIAL CASE OF

FULCRUM NUMBERS

BECOMES THERAMANUJAN NUMBERS

Eq. 3) $2B(J-K) = K^2+K+J^2+J$

when $J = B-1$

3) becomes

$$B^2 = B(2K+1)+K^2+K$$

1, 2, 3, 4, 5 |6| 7, 8

5 2

$\boxed{\Delta = 3}$

Σ 15 Σ 15

SQUARES

$\Delta = 1$

1, 2 ····34 |35| 36 ····49

34 13

$\boxed{\Delta = 21}$

Σ 595 Σ 595

1, 2, ··· 203 |204| 205 ···· 288

203 83

$\boxed{\Delta = 120}$

Σ = 20706 Σ 20706

1, 2, ---- 1188 |1189| 1190 --- 1681

1188 491

$\boxed{\Delta = 697}$

Σ = 706266 Σ 706266

$$K = \frac{\sqrt{1+8B^2} - 1}{2} - B$$

$B = 1, \quad K = 0$ ✓

$6, \quad K = 2$ ✓

$$\Sigma = \sum_{1}^{B-1} i = \frac{B(B-1)}{2}$$

B	Σ
6	15
35	595

$$B = \frac{\sqrt{8K^2+8K+1}}{2} + K + \frac{1}{2}$$

$K = 0, \quad B = 1$

$K = 2, \quad B = 6$

RAMANUJAN NUMBERS

There are many stories recalling the mathematical genius of the Indian mathematician, Srinivasa Ramanujan. One of these was during World War One while Ramanujan lived in England. He was told about a man who had a friend who lived in Louvain, a city recently destroyed by the invading Germans. His friend lived in a house whose number he could not remember but he did remember that his friend told him:

> "He lived on a street in which all of the houses on his side of the street were numbered one, two, three, and so on. He said that the sum of the numbers of the houses on the left side of his house was exactly the same as the sum of the numbers on the right side. He knew there were more than 50 houses on his side of the street, but less than 300. [1]

Would it be possible to reconstruct the number of the house from this information?

When Ramanujan heard the story he immediately said the number of the house was 204. How could you tell that he was asked.

---

[1] This story is recounted in Robert Kanigel's book, "The Man who knew Infinity".

The popular English magazine *Strand* had long carried a page, entitled "Perplexities," devoted to intriguing puzzles, numbered and charmingly titled, like "The Fly and the Honey," or "The Tessellated Tiles," the answers being furnished the following month. Each Christmas, though, "Perplexities" expanded, the author fitting his puzzles into a short story. Now, in December 1914, "Puzzles at a Village Inn" took readers to the imaginary town of Little Wurzelfold, where the main topic of interest was what had just happened in Louvain.

In late August, pursuing an explicit policy of brutalization against civilian populations, German troops began burning the medieval Belgian city of Louvain, on the road between Liège and Brussels. House by house and street by street they set Louvain to the torch, destroying its great library, with its quarter million books and medieval manuscripts, and killing many civilians. The burning of Louvain horrified the world, galvanized public opinion against Germany, and united France, Russia, and England more irrevocably yet. "The March of the Hun," English newspapers declared. "Treason to Civilization." It was an early turning point of the war, doing much to set its tone. *Louvain* came to symbolize the breakdown of civilization. And now it reached even the "Perplexities" page of *Strand.*

One Sunday morning soon after the December issue appeared, P. C. Mahalanobis sat with it at a table in Ramanujan's rooms in Whewell's Court. Mahalanobis was the King's College student, just then preparing for the natural sciences Tripos, who had found Ramanujan shivering by the fireplace and schooled him in the nuances of the English blanket. Now, with Ramanujan in the little back room stirring vegetables over the gas fire, Mahalanobis grew intrigued by the problem and figured he'd try it out on his friend.

"Now here's a problem for you," he yelled into the next room

"What problem? Tell me," said Ramanujan, still stirring. And Mahalanobis read it to him.

> "I was talking the other day," said William Rogers to the other villagers gathered around the inn fire, "to a gentleman about the place called Louvain, what the Germans have burnt down. He said he knowed it well—used to visit a Belgian friend there. He said the house of his friend was in a long street, numbered on this side one, two, three, and so on, and that all the numbers on one side of him added up exactly the same as all the numbers on the other side of him. Funny thing that! He said he knew there was more than fifty houses on that side of the street, but not so many as five hundred. I made mention of the matter to our parson, and he took a pencil and worked out the number of the house where the Belgian lived. I don't know how he done it."
>
> Perhaps the reader may like to discover the number of that house.

Through trial and error, Mahalanobis (who would go on to found the Indian Statistical Institute and become a Fellow of the Royal Society) had figured it out in a few minutes. Ramanujan figured it out, too, but with a twist. "Please take down the solution," he said—and proceeded to dictate a continued fraction, a fraction whose denominator consists of a number plus a fraction, *that* fraction's denominator consisting of a number plus a fraction, ad infinitum. This wasn't just the solution to the problem, it was the solution to the whole class of problems implicit in the puzzle. As stated, the problem had but one solution—house no. 204 in a street of 288 houses; $1+2+\ldots 203 = 205+206+\ldots 288$. But without the 50-to-500 house constraint, there were other solutions. For example, on an eight-house street, no. 6 would be the answer: $1+2+3+4+5$ on its left equaled $7+8$ on its right. Ramanujan's continued fraction comprised within a single expression *all* the correct answers.

Mahalonobis was astounded. How, he asked Ramanujan, had he done it?

"Immediately I heard the problem it was clear that the solution should obviously be a continued fraction; I then thought, Which continued fraction? And the answer came to my mind."

4. THE ZEROES OF THE ZETA FUNCTION

The answer came to my mind. That was the glory of Ramanujan—that so much came to him so readily, whether through the divine offices of the goddess Namagiri, as he sometimes said, or through what Westerners might ascribe, with equal imprecision, to "intuition." And yet, it was the very power of his intuition that, in one sense, undermined his mathematical development. For it blinded him to intuition's limits, gave him less reason to learn modern mathematical tools, shielded him from his own ignorance.

"The limitations of his knowledge were as startling as its profundity," Hardy would write.

RAMINTRO.WPD ✓ January 8, 2004

RAMANUJAN NUMBERS

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> "He lived on a street in which all of the houses on his side of the street were numbered one, two, three, and so on. He said that the sum of the numbers of the houses on the left side of his house was exactly the same as the sum of the numbers on the right side. He knew there were more than 50 houses on his side of the street, but less than 300. [1]

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---

[1] This story is recounted in Robert Kanigel's book, "The Man who knew Infinity".

LOUVAIN.WPD ✓ February 18, 2004

From the book, THE MAN WHO KNEW INFINITY, A life of the genius Ramanujan by Robert Kanigel Washington Square Press 1991

The following on page 214-215 is the origin of the house number problem:

> "I was talking the other day," said William Rogers to the other villagers gathered around the inn fire, "to a gentleman about the place called Louvain, what the Germans have burnt down. He said he knowed it well–used to visit a Belgian friend there. He said the house of his friend was in a long street, numbered on this side one, two, three, and so on, and that all the numbers on one side of him added up exactly the same as all the numbers on the other side of him. Funny thing that! He said he knew there was more than fifty houses on that side of the street, but not so many as five hundred. I made mention of the matter to our parson, and he took a pencil and worked out the number of the house where the Belgian lived. I don't know how he done it."

Perhaps the reader may like to discover the number of that house.

When this puzzle was told to Ramanujan, he immediately came up with the answer:
The number of the house was 204. And Ramanujan went further and gave a formula for all numbers having the property that the sum of all numbers less than the fulcrum number had an equal integer sequence above.

The book does not give any further info. But both a recursion and an explicit formula for all such fulcrum numbers can readily be derived. The formulae are quite similar to those for the Fibonacci sequence, but which AOL does not permit to be communicated.

RAMANUJ SEQUENCES

$B(n+2) = 6B(n+1) - B(n)$ [0,1] $D(n+2) = 6D(n+1) - D(n)$ [1,1]

	56				
Bσ2	8, 48, 280, 1632, 9512, 55440	8B	8, 40, 232, 1352, 7880, 45928	8D	Dσ2
Bσ1	1, 7, 41, 239, 1393, 8119, 47321	αK	2, 6, 34, 198, 1154, 6726 39202	α2K αKσ2	Dσ1
B(n)	0, 1, 6, 35, 204, 1189, 6930, 40391		1, 1, 5, 29, 169, 985, 5741, 33461	BΔ1	D(n)
BΔ1	1, 5, 29, 169, 985, 5741, 33461,	D	0, 4, 24, 140, 816, 4756, 27720	α WΔ2 α 2W 4B, BΔ2	DΔ1
BΔ2	4, 24, 140, 816, 4756, 27720	DΔ1, 4B	4, 20, 116, 676, 3940, 22964	4D, 4BΔ1	DΔ2

Wσ2	4 10 24 58 140 338 816	2W	6, 14, 34, 82, 198, 478, 1154	2K	Kσ2
Wσ1	1, 3, 7, 17, 41, 99, 239, 577	K	2, 4, 10, 24, 58, 140, 338, 816	2W Wσ2	Kσ1
W(n)	0, 1, 2, 5, 12, 29, 70, 169, 408, 985		1, 1, 3, 7, 17, 41, 99, 239, 577		K(n)
WΔ1	1, 1, 3, 7, 17, 41, 99, 239	K	0, 2, 4, 10, 24, 58, 140, 338	2W	KΔ1
WΔ2	0, 2, 4, 10, 24, 58, 140	2W	2, 2, 6, 14, 34, 82, 198	2K	KΔ2

$W(n+2) = 2W(n+1) + W(n)$ [0,1] $K(n+2) = 2K(n+1) + K(n)$ [1,1]

CODE
σ SUM
Δ DIFF
α alternate

$W(n) \cdot K(n) = B(n)$

$K(n) = W\Delta 1(n)$ $\quad$ $D(n) = B\Delta 1(n-1)$

$K(n) = W\delta 1(n-1)$

$K\delta i = K\Delta i$ all i

$B(n+2) - B(n) = K\delta 2(2n-1) = 2K(2n-1)$

$H\ G\ J\ \Sigma$ $\quad$ $\sqrt{G} = \alpha K$

$U = H - B$ $\quad$ $\sqrt{J} = \mathring{\alpha} K$

$W(n) \cdot K(n) = B(n)$

$W(n) \cdot W\delta 1(n-1) = B(n)$

$W(n) \cdot W\Delta 1(n) = B(n)$

RAMANU N SEQUENCES

$B(n+2) - B(n) = 2K(2n+1)$

$B(n+1) - B(n-1) = 2 \cdot G(n)$

$E(n) = 2B(n) - B(n-1)$

$A(n+2) = 6A(n+1) - A_n$

			0, 1, 6, 35, 204, 1189, 6930, 40391,		B	[0,1]
	460		1, 1, 5, 29, 169, 985, 5741, 33461	alt W	D	[1,1]
	66 394		1, 3, 17, 99, 577, 3363	alt K	G	[1,3]
	9 57 337 1969 11681		0, 2, 12, 70, 408, 2378	2B		[0,2]
H(n)	1 8 49 288 1681 9800 57121 332928		0, 3, 18, 105, 612, 3567	3B		[0,3]
HΔ1	7 41 239 1393	Bσ1	1, 2, 11, 64, 373, 2174		E	[1,2]
	34 198 1154					
			0, 1, 2, 5, 12, 29, 70, 169, 408, 985	alt 2B	W	[0,1]
			1, 1, 3, 7, 17, 41, 99, 239, 577		K	[1,1]
	18 104		1, 3, 7, 17, 41, 99	K		[1,3]
Uσ1	2 16 98 576	2H(n)	0, 2, 4, 10, 24, 58, 130, 318, 766			[0,2]
U(n)	0 2 14 84 492 2870 16730 97512 568334		0, 3, 6, 15, 36, 87, 200, 487,		[illegible]	[0,3]
UΔ1	2 12 70 408 2378	2B(n)	1, 2, 5, 12, 29, 70, 169, 408, 985	Alt 2B	W	[1,2]
	10 58					

$A(n+2) = 2A(n+1) + A(n)$

$H(n) - B\sigma 2(n) = H(n-2)$

$\frac{U(n)}{2} = B(n)$

$U(n) = H(n) - B(n)$

$H_{n+1} - H_{n-1} = 8B_n$

RAMRHOMB.WPD

RAMAJUNAN NUMBERS RHOMBOID

January 8, 2004

σ

9

Σ9										33255424									
Σ8									4870144		28385280								
Σ7								713216		4156928		24228352							
Σ6							104448		608768		3548160		20680192						
Σ5						15296		89152		519616		3028544		17651648					
Σ4					2240		13056		76046		443520		2585024		15066624				
Σ3				328		1912		11144		64952		378568		2206456		12860168			
Σ2			48		280		1632		9512		55440		323128		1883328		10976840		
Σ1		7		41		239		1393		8119		47321		275807		1607521		9369319	
B	1		6		35		204		1189		6930		40391		235416		1372105		7997214
Δ1		5		29		169		985		5741		33461		195025		1136689		6625109	
Δ2			24		140		816		4756		27720		161564		941664		5488420		
Δ3				116		676		3940		22964		133844		780100		4546756			
Δ4					560		3264		19024		110880		646256		3766656				
Δ5						2704		15760		91856		535376		3120400					
Δ6							13056		76096		443520		2585024						
Δ7								63040		367424		2141504							
Δ8									304384		1774080								
Δ9										1469696									

$4\cdot\Delta_1 = \Delta_3$
$4\cdot\Delta_2 = \Delta_4$
$4\cdot\Delta_3 = \Delta_5$
$4\cdot\Delta_4 = \Delta_6$
$4\cdot\Delta_5 = \Delta_7$
$4\cdot\Delta_6 = \Delta_8$
$4\cdot\Delta_7 = \Delta_9$
$\cdots$
$4\cdot\Delta_n = \Delta_{n+2}$

$8\cdot\Sigma_1 = \Sigma_3$
$8\cdot\Sigma_2 = \Sigma_4$
$8\cdot\Sigma_3 = \Sigma_5$
$8\cdot\Sigma_4 = \Sigma_6$
$8\cdot\Sigma_5 = \Sigma_7$
$8\cdot\Sigma_6 = \Sigma_8$
$8\cdot\Sigma_7 = \Sigma_9$
$\cdots$
$8\cdot\Sigma_n = \Sigma_{n+2}$

$2\cdot\Delta_2 = \Sigma_2$
[illegible]$\cdot\Delta_4 = 2\Sigma_2$
$4\cdot\Delta_4 = \Sigma_4$
$\boxed{\Delta_6 = \Sigma_4}$
$\Delta_8 = 4\Sigma_4$
$2\cdot\Delta_8 = \Sigma_6$
$8\cdot\Delta_6 = \Sigma_6$

~~$2^n\Delta_{2n} = \Sigma_{2m}$~~
~~Σ_{2m}~~
$2^n\Delta_{2n} = \sigma_{2n}$?

Σ should be replaced with σ

$8B = \sigma_2$
$64B = \sigma_4$
~~$n^3 B = \sigma_n$~~
$4B = \Delta_2$
$16B = \Delta_4$
$\boxed{2^n B = \Delta_n \text{ for } n \text{ even}}$

$2^{\cancel{3/2\,n}} B = \sigma_n$ n even
$\boxed{2^{\frac{3}{2}n} B = \sigma_n \text{ even}}$

All σ's and Δ's obey $\Delta(n+2) = 6\,\Delta(n+1) - \Delta(n)$

RAMAUJAN SEQUENCES

Notation original ~~Entry~~

σ_i and derived types

Δ_i

Σ sum

α = alt even

β = alt odd

$r(n) = \sqrt{g(n)} = 7B(n) - B(n-1) = B(n)+B(n+1)$

B[0,1]	B(n)	1	6	35	204	1189	6930	40391	235416	1372105	B
D[1,1]	$\Delta 1 B =$	ALTW	5	29	169	985	5741	33461	195025	1136689	alt W
E	$\sigma 1 B =$	ALT σ1W (	7	41	239	1393	8119	47321	275807	1607521	alt K
W [0,1]	W =	2	5	12	29	70	169	408	985		W
K[1,1]	$\sigma 1 W =$ ΔIW	1	3	7	17	41	99	239	577		K
H	~~Higead #~~	1	8	49	288	1681	9800	57121	332928	1940449	H
U = H−B		0	2	14	84	492	2870	16730	97512	568344	H−B
Σ	$B\cdot(B-1)/2$	0	15	595	20706	706266	24008985				Σ
G	$1+8B^{2}(n)^{2}$	9	289	9801	332929	11309769					$(\text{alt } K)^2$
R(n) $\sqrt{G}$		3	17	99	577	3363	19601	114243			$\overline{\text{alt}}$ K
J g(n)		1	49	1681	57121						$(\text{alt } K)^2$
r(n) $\sqrt{g(n)}$		1	7	41	239	1393	8119	47321	275807	1607521	alt K
	= 2B	2	12	70	408	2378					$\overline{\text{alt}}$W

$A(n+2) = 6\cdot A(n+1) - A(n)$

for B, D, σ1B, E

$A(n+2) = 2\cdot A(n+1) + A(n))$

For W, σ1W, K

$\sigma_1 W(n) \times W(n-1) = B(n)$

$K(n) \times W(n-1) = B(n)$

$\Delta_2 B = \Delta 1 W = 4B$

$\sigma_2 K = 2K$

$\mathfrak{J}(n) = \frac{\sqrt{1+8B(2n)^2}-1}{2}$

$\Delta 1 B(n) + \sigma 1 B(n) = 2B(n)$

$\Delta 1 B(n) - \sigma 1 B(n) = 2B(n-1)$

check

$H_{n+1} - H_{n-1} = 8B_n$

$B\sigma 2(n) + H(n-2) = H(n)$

1° Recursion Formula
2° Initials
4° Explicit Formulas
3° σ's and Δ's
4

Ramanujans

$B(n+2) = 6\,B(n+1) - B(n)$ $[0,1] = B$'s
$[1,1] = D$'s
$[1,3] = A1+K$
$A(n+2) = 2\,A(n+1) + A(n)$ $[0,1] = W$'s
$[1,1] = R$'s

APHSYMB.WPD June 20, 2006

APHORISMS RE SEMIOTICS

The affective structures of the human being, though unconscious, are expressed in words, fantasies, metaphors, dreams, and symptoms. Clearly these are not structures of behavior; they are closer to what others call cognitive structures or primitive beliefs.

Words both express and condition thought

Language by its nature tends to distort experience.
–Joyce Carol Oates

Symbols participate in the ~~world~~ reality they represent.
–Paul Tillich

The symbol has meaning which transcends the object symbolized. [and transforms]
–Tobias Dantzig

What we already know
direct what we can learn
discover
and governs what we can think
[act how]

Feynman "I knew too much"

A word is the abstract symbol of a class, yet it also has the capacity to evoke an image, a concrete picture of some representative element of the class.
–Tobias Dantzig

Disparate objects can, by the use of abstraction, be seen to be visually related. –Howard Steinberg

In attempt to make experience intelligible, analogy [or metaphor] plays a fundamental role. By means of it what is already familiar or understood is appealed to in order to make clear the unfamiliar and unexplained. {This works because of the redundant and fractal and recursive nature of the world.)
–Munitz

THOSE SOCIETIES WHICH CANNOT COMBINE REVERENCE TO THEIR SYMBOLS WITH FREEDOM OF REVISION, MUST ULTIMATELY DECAY EITHER FROM ANARCHY, OR FROM THE SLOW ATROPHY OF A LIFE STIFLED BY USELESS SHADOWS. –A.N.WHITEHEAD

Rhetoric rather than truth is crucial for carrying an argument.
–WHITEHEAD

We begin to understand an inherent ethical catch in the new technical order, its obligation to rely on the misuse of symbols.

There is a hopeful symbolism in the fact that flags will not wave in a vacuum. –Arthur C.Clark

The bomb is a symbol for the worst of modernity.
–Spencer Weart

RAMNUMB.MCD JUNE 23, 2006

RAMANUJAN NUMBERS RECUSION FORMULA: B(n+2) = 6 B(n+1) - B(n)

$p := 3 + \sqrt{8}$ $\qquad$ $q := 3 - \sqrt{8}$

$n := 0, 1 .. 15$

$$B(n) := \frac{\left(p^{n+1} - q^{n+1}\right)}{2 \cdot \sqrt{8}} \qquad H(n) := \frac{\left(-1 + \sqrt{1 + 8 \cdot B(n)^2}\right)}{2}$$

$$S(n) := B(n) \cdot \frac{(B(n) - 1)}{2} \qquad U(n) := H(n) - B(n) \qquad L(n) := B(n) - 1$$

B = Ramanujan number, H = highest number, S = summation

n =	B(n) =	H(n) =	U(n) =	S(n) =
0	1	1	0	0
1	6	8	2	15
2	35	49	14	595
3	204	288	84	20706
4	1189	1681	492	706266
5	6930	9800	2870	24008985
6	40391	57121	16730	815696245
7	235416	332928	97512	27710228820
8	1372105	1940449	568344	$9.413 \cdot 10^{11}$
9	7997214	11309768	3312554	$3.198 \cdot 10^{13}$
10	46611179	65918161	19306982	$1.086 \cdot 10^{15}$
11	271669860	384199200	112529340	$3.69 \cdot 10^{16}$
12	1583407981	2239277041	655869060	$1.254 \cdot 10^{18}$
13	9228778026	13051463048	3822685022	$4.259 \cdot 10^{19}$
14	53789260175	76069501249	22280241074	$1.447 \cdot 10^{21}$
15	$3.135 \cdot 10^{11}$	$4.434 \cdot 10^{11}$	$1.299 \cdot 10^{11}$	$4.914 \cdot 10^{22}$

The even terms of H(n) are perfect squares, the odd terms + 1 are perfect squares

RAMNUMB2

$$p := 3 + \sqrt{8} \qquad q := 3 - \sqrt{8}$$

$$n := 0, 1 .. 9$$

$$B(n) := \frac{\left(p^{n+1} - q^{n+1}\right)}{2 \cdot \sqrt{8}} \qquad G(n) := 1 + 8 \cdot B(n)^2 \qquad R(n) := \sqrt{G(n)}$$

$$H_{2n}\,(\text{[illegible]}) \equiv g(n) := \frac{\left(-1 + \sqrt{1 + 8 \cdot B(2 \cdot n)^2}\right)}{2} \qquad r(n) := \sqrt{g(n)}$$

n =	B(n) =	G(n) =	R(n) =	g(n) = H(2n)	r(n) =
0	1	9	3	1	1
1	6	289	17	49	7
2	35	9801	99	1681	41
3	204	332929	577	57121	239
4	1189	11309769	3363	1940449	1393
5	6930	384199201	19601	65918161	8119
6	40391	13051463049	114243	2239277041	47321
7	235416	$4.434 \cdot 10^{11}$	665857	76069501249	275807
8	1372105	$1.506 \cdot 10^{13}$	3880899	$2.584 \cdot 10^{12}$	1607521
9	7997214	$5.116 \cdot 10^{14}$	22619537	$8.778 \cdot 10^{13}$	9369319

The G(n) and g(n) are perfect squares and are therefore also fulcrum numbers

RANNUMB3 ✓

$p := 3 + \sqrt{8}$ $\quad q := 3 - \sqrt{8}$ $\quad P := 1 + \sqrt{2}$ $\quad Q := 1 - \sqrt{2}$

$n := 0, 1 .. 10$

$$B(n) := \frac{\left(p^{n+1} - q^{n+1}\right)}{2 \cdot \sqrt{8}} \qquad g(n) := \frac{\left(-1 + \sqrt{1 + 8 \cdot B(2 \cdot n)^2}\right)}{2} \qquad r(n) := \sqrt{g(n)}$$

$$u(n) := 10 \cdot \frac{\left(p^{n-1} - q^{n-1}\right)}{\sqrt{8}} \qquad v(n) := 7 \cdot \frac{\left(p^{n-1} + q^{n-1}\right)}{2} \qquad D(n) := u(n) + v(n)$$

$$E(n) := 7 \cdot B(n) - B(n-1) \qquad J(n) := B(n) + B(n+1)$$

n =	E(n) =	J(n) =	r(n) =	D(n) =
0	7	7	1	1
1	41	41	7	7
2	239	239	41	41
3	1393	1393	239	239
4	8119	8119	1393	1393
5	47321	47321	8119	8119
6	275807	275807	47321	47321
7	1607521	1607521	275807	275807
8	9369319	9369319	1607521	1607521
9	54608393	54608393	9369319	9369319
10	318281039	318281039	54608393	54608393

RAMNUMB4

$p := 3+\sqrt{8}$ $\quad q := 3-\sqrt{8}$ $\quad P := 1+\sqrt{2}$ $\quad Q := 1-\sqrt{2}$

$n := 0, 1 .. 10$

$$B(n) := \frac{\left(p^{n+1} - q^{n+1}\right)}{2\cdot\sqrt{8}} \qquad G(n) := 8\cdot B(n)^2 + 1 \qquad R(n) := \sqrt{G(n)}$$

$$g(n) := \frac{\left(-1+\sqrt{1+8\cdot B(2\cdot n)^2}\right)}{2} \qquad r(n) := \sqrt{g(n)} \qquad K(n) := \frac{\left(P^n + Q^n\right)}{2}$$

n =	B(n) =	R(n) =	r(n) =	K(n) =
0	1	3	1	1
1	6	17	7	1
2	35	99	41	3
3	204	577	239	7
4	1189	3363	1393	17
5	6930	19601	8119	41
6	40391	114243	47321	99
7	235416	665857	275807	239
8	1372105	3880899	1607521	577
9	7997214	22619537	9369319	1393
10	46611179	131836323	54608393	3363

RAMNUMB5 ✓

Sequence: 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601
is the merging of R(n) and r(n) preserving numerical size.

Recursion formula: K(n+2) = 2K(n+1)+K(n)

$p := 1 + \sqrt{2}$ $\qquad$ $q := 1 - \sqrt{2}$

Explicit formula:

$$K(n) := \frac{\left(p^n + q^n\right)}{2}$$

$n := 1, 2 .. 15$

n =	K(n) =
1	1
2	3
3	7
4	17
5	41
6	99
7	239
8	577
9	1393
10	3363
11	8119
12	19601
13	47321
14	114243
15	275807

✓

RAMANOS.MCD

RAMANUJAN NUMBERS
B(n) and their sums
Σ1,Σ3,Σ5,Σ7,Σ9

January 8, 2004

Explicit Formulae

Recursion Formula

B(n) = 6 B(n-1) - B(n-2)

$p := 3 + \sqrt{8}$ $\qquad$ $q := 3 - \sqrt{8}$

$$B(n) := \frac{\left(p^n - q^n\right)}{2\sqrt{8}}$$

$\Sigma 1(n) := 7 \cdot B(n) - B(n-1)$ $\qquad$ $\Sigma 3(n) := 8 \cdot \Sigma 1(n)$ $\qquad$ $\Sigma 5(n) := 8 \cdot \Sigma 3(n)$

$n := 1, 2 .. 10$

n =	B(n) =	Σ1(n) =	Σ3(n) =
1	1	7	56
2	6	41	328
3	35	239	1912
4	204	1393	11144
5	1189	8119	64952
6	6930	47321	378568
7	40391	275807	2206456
8	235416	1607521	12860168
9	1372105	9369319	74954552
10	7997214	54608393	436867144

$\Sigma 7(n) := 8 \cdot \Sigma 5(n)$ $\qquad$ $\Sigma 9(n) := 8 \cdot \Sigma 7(n)$

n =	Σ5(n) =	Σ7(n) =	Σ9(n) =
1	448	3584	28672
2	2624	20992	167936
3	15296	122368	978944
4	89152	713216	5705728
5	519616	4156928	33255424
6	3028544	24228352	193826816
7	17651648	141213184	$1.1297055 \cdot 10^{9}$
8	102881344	823050752	$6.584406 \cdot 10^{9}$
9	599636416	$4.7970913 \cdot 10^{9}$	$3.8376731 \cdot 10^{10}$
10	$3.4949372 \cdot 10^{9}$	$2.7959497 \cdot 10^{10}$	$2.2367598 \cdot 10^{11}$

RAMANES.MCD

RAMANUJAN NUMBERS
B(n) and their sums
$\Sigma 2, \Sigma 4, \Sigma 6, \Sigma 8, \Sigma 10$

January 8, 2004

Explicit Formulae

Recursion Formula

B(n) = 6 B(n-1) - B(n-2)

$p := 3 + \sqrt{8}$ $\qquad$ $q := 3 - \sqrt{8}$

Σ

$$B(n) := \frac{\left(p^n - q^n\right)}{2\sqrt{8}}$$

$\Sigma 2(n) := 8 \cdot B(n)$ $\qquad$ $\Sigma 4(n) := 8 \cdot \Sigma 2(n)$ $\qquad$ $\Sigma 6(n) := 8 \cdot \Sigma 4(n)$

$n := 1, 2 .. 10$

n =	B(n) =	Σ2(n) =	Σ4(n) =
1	1	8	64
2	6	48	384
3	35	280	2240
4	204	1632	13056
5	1189	9512	76096
6	6930	55440	443520
7	40391	323128	2585024
8	235416	1883328	15066624
9	1372105	10976840	87814720
10	7997214	63977712	$5.11821696 \cdot 10^{8}$

$\Sigma 8(n) := 8 \cdot \Sigma 6(n)$ $\qquad$ $\Sigma 10(n) := 8 \cdot \Sigma 8(n)$

n =	Σ6(n) =	Σ8(n) =	Σ10(n) =
1	512	4096	32768
2	3072	24576	196608
3	17920	143360	1146880
4	104448	835584	6684672
5	608768	4870144	38961152
6	3548160	28385280	$2.2708224 \cdot 10^{8}$
7	20680192	$1.65441536 \cdot 10^{8}$	$1.323532288 \cdot 10^{9}$
8	$1.20532992 \cdot 10^{8}$	$9.64263936 \cdot 10^{8}$	$7.714111488 \cdot 10^{9}$
9	$7.0251776 \cdot 10^{8}$	$5.62014208 \cdot 10^{9}$	$4.496113664 \cdot 10^{10}$
10	$4.094573568 \cdot 10^{9}$	$3.275658854 \cdot 10^{10}$	$2.620527084 \cdot 10^{11}$

RAMANOD.MCD

RAMANUJAN NUMBERS
B(n) and their differences
Δ1,Δ3,Δ5,Δ7,Δ9

January 8, 2004

Explicit Formulae

Recursion Formula

B(n) = 6 B(n-1) - B(n-2)

$p := 3 + \sqrt{8}$ $\qquad$ $q := 3 - \sqrt{8}$

$$B(n) := \frac{\left(p^n - q^n\right)}{2\sqrt{8}}$$

$\Delta1(n) := 5 \cdot B(n-1) - B(n-2)$ $\qquad$ $\Delta3(n) := 4 \cdot \Delta1(n)$ $\qquad$ $\Delta5(n) := 4 \cdot \Delta3(n)$

$n := 1, 2 .. 10$

n =	B(n) =	Δ1(n) =	Δ3(n) =
1	1	1	4
2	6	5	20
3	35	29	116
4	204	169	676
5	1189	985	3940
6	6930	5741	22964
7	40391	33461	133844
8	235416	195025	780100
9	1372105	1136689	4546756
10	7997214	6625109	$2.6500436 \cdot 10^7$

$\Delta7(n) := 4 \cdot \Delta5(n)$ $\qquad$ $\Delta9(n) := 4 \cdot \Delta7(n)$

n =	Δ5(n) =	Δ7(n) =	Δ9(n) =
1	16	64	256
2	80	320	1280
3	464	1856	7424
4	2704	10816	43264
5	15760	63040	252160
6	91856	367424	1469696
7	535376	2141504	8566016
8	3120400	$1.24816 \cdot 10^7$	$4.99264 \cdot 10^7$
9	$1.8187024 \cdot 10^7$	$7.2748096 \cdot 10^7$	$2.9099238 \cdot 10^8$
10	$1.0600174 \cdot 10^8$	$4.2400698 \cdot 10^8$	$1.6960279 \cdot 10^9$

RAMANED.MCD

RAMANUJAN NUMBERS
B(n) and their differences
$\Delta 2, \Delta 4, \Delta 6, \Delta 8, \Delta 10$

January 8, 2004

Explicit Formulae

Recursion Formula

$B(n) = 6\,B(n-1) - B(n-2)$

$p := 3 + \sqrt{8}$ $\qquad q := 3 - \sqrt{8}$

$$B(n) := \frac{\left(p^{n} - q^{n}\right)}{2\sqrt{8}}$$

$\Delta 2(n) := 4 \cdot B(n)$ $\qquad \Delta 4(n) := 4 \cdot \Delta 2(n)$ $\qquad \Delta 6(n) := 4 \cdot \Delta 4(n)$

$n := 1, 2 .. 10$

n =	B(n) =	Δ2(n) =	Δ4(n) =
1	1	4	16
2	6	24	96
3	35	140	560
4	204	816	3264
5	1189	4756	19024
6	6930	27720	110880
7	40391	161564	646256
8	235416	941664	3766656
9	1372105	5488420	$2.195368\cdot 10^{7}$
10	7997214	$3.1988856\cdot 10^{7}$	$1.2795542\cdot 10^{8}$

$\Delta 8(n) := 4 \cdot \Delta 6(n)$ $\qquad \Delta 10(n) := 4 \cdot \Delta 8(n)$

n =	Δ6(n) =	Δ8(n) =	Δ10(n) =
1	64	256	1024
2	384	1536	6144
3	2240	8960	35840
4	13056	52224	208896
5	76096	304384	1217536
6	443520	1774080	7096320
7	2585024	$1.0340096\cdot 10^{7}$	$4.1360384\cdot 10^{7}$
8	$1.5066624\cdot 10^{7}$	$6.0266496\cdot 10^{7}$	$2.4106598\cdot 10^{8}$
9	$8.781472\cdot 10^{7}$	$3.5125888\cdot 10^{8}$	$1.4050355\cdot 10^{9}$
10	$5.118217\cdot 10^{8}$	$2.0472868\cdot 10^{9}$	$8.1891471\cdot 10^{9}$

RAMANUMB26

$p := 3 + \sqrt{8}$

$s := \dfrac{p^2}{p^2 - 1}$

$B(n) := p^n \cdot s$

$n := 0, 1 .. 12$

n =	B(n) =	round(B(n)) =
0	1.03	1
1	6.005	6
2	35.001	35
3	204	204
4	1189	1189
5	6930	6930
6	40391	40391
7	235416	235416
8	1372105	1372105
9	7997214	7997214
10	46611179	46611179
11	271669860	271669860
12	$1.583 \cdot 10^9$	$1.583 \cdot 10^9$

$$X(n) = \frac{-1 + \sqrt{1 + \frac{\cancel{8}(p^{n+1} - q^{n+1})^2}{4 \cdot \cancel{8}}}}{2}$$

$$-\frac{1}{2} + \frac{1}{2}\sqrt{1 + (p^{n+1} - q^{n+1})^2}$$

$$X(n) = \frac{\sqrt{1 + (p^{n+1} - q^{n+1})^2}}{2} - \frac{1}{2}$$

RAMANex1.MCD

RAMANUJAN NUMBERS
B(n)

January 7, 2004

Explicit Formulae

$p := 3 + \sqrt{8}$ $q := 3 - \sqrt{8}$

Recursion Formula

B(n) = 6 B(n-1) - B(n-2)

$$B(n) := \frac{p^n}{2\sqrt{8}} \qquad b(n) := \frac{\sqrt{p^n}}{2\sqrt{8}} \qquad c(n) := \frac{p^{\frac{n}{3}}}{2\sqrt{8}} \qquad d(n) := \frac{p^{\frac{n}{4}}}{2\sqrt{8}}$$

$n := 1, 2 .. 18$

n =	round(B(n)) =	round(b(n)) =	round(c(n)) =	round(d(n)) =
1	1	0	0	0
2	6	1	1	0
3	35	2	1	1
4	204	6	2	1
5	1189	14	3	2
6	6930	35	6	2
7	40391	84	11	4
8	235416	204	19	6
9	1372105	492	35	9
10	7997214	1189	63	14
11	46611179	2870	113	23
12	271669860	6930	204	35
13	$1.583407981\cdot10^{9}$	16730	367	54
14	$9.228778026\cdot10^{9}$	40391	661	84
15	$5.378926018\cdot10^{10}$	97512	1189	131
16	$3.13506783\cdot10^{11}$	235416	2140	204
17	$1.827251438\cdot10^{12}$	568344	3851	317
18	$1.065000184\cdot10^{13}$	1372105	6930	492

$$S := \frac{1}{2\sqrt{8}}$$

$p = 5.828427125$

$q = 0.171572875$

$p + q = 6$

Roots insert numbers

$$S := \frac{p}{p^2 - 1}$$

$S = 0.176776695$

$p^{\frac{n}{\alpha}}$ inserts $\alpha - 1$

RAMANex2.MCD RAMANUJAN NUMBERS January 7, 2004

B(n)

Explicit Formulae

$p := 3 + \sqrt{8}$ $q := 3 - \sqrt{8}$

Recursion Formula

B(n) = 6 B(n-1) - B(n-2)

$B(n) := \frac{p^{n}}{2\sqrt{8}}$ $b(n) := \frac{p^{2n}}{2\sqrt{8}}$ $c(n) := \frac{p^{3n}}{2\sqrt{8}}$

$n := 1, 2 .. 15$

n =	round(B(n)) =	round(b(n)) =	round(c(n)) =
1	1	6	35
2	6	204	6930
3	35	6930	1372105
4	204	235416	$2.7166986 \cdot 10^{8}$
5	1189	7997214	$5.378926018 \cdot 10^{10}$
6	6930	$2.7166986 \cdot 10^{8}$	$1.065000184 \cdot 10^{13}$
7	40391	$9.228778026 \cdot 10^{9}$	$2.108646576 \cdot 10^{15}$
8	235416	$3.13506783 \cdot 10^{11}$	$4.17501372 \cdot 10^{17}$
9	1372105	$1.065000184 \cdot 10^{13}$	$8.266316302 \cdot 10^{19}$
10	7997214	$3.617865559 \cdot 10^{14}$	$1.636688878 \cdot 10^{22}$
11	$4.6611179 \cdot 10^{7}$	$1.22900929 \cdot 10^{16}$	$3.240561315 \cdot 10^{24}$
12	$2.7166986 \cdot 10^{8}$	$4.17501372 \cdot 10^{17}$	$6.416147734 \cdot 10^{26}$
13	$1.583407981 \cdot 10^{9}$	$1.418275656 \cdot 10^{19}$	$1.270364846 \cdot 10^{29}$
14	$9.228778026 \cdot 10^{9}$	$4.817962216 \cdot 10^{20}$	$2.515258233 \cdot 10^{31}$
15	$5.378926018 \cdot 10^{10}$	$1.636688878 \cdot 10^{22}$	$4.980084265 \cdot 10^{33}$

$S := \frac{1}{2\sqrt{8}}$

$p = 5.828427125$

$p + q = 6$

$q = 0.171572875$

$S := \frac{p}{p^{2} - 1}$

$S = 0.176776695$

Powers skip numbers

$p^{\alpha n}$ skips $\alpha - 1$

$F_0 = 1$

$F_1 = 6$

$F_2 = 35$

$F_3 = 204$

$F_4 = 1189$

$F_5 = 6930$

$F_6 = 40391$

$F_7 = 235416$

$F_8 = 1372105$

$F_9 = 7997214$

$F_{10} = 46611179$

RECURSION FORMULA $F_N = 6\,F_{N-1} - F_{N-2}$

NOTE LAST FIGURE SEQUENCE
1 6 5 4 9 0

~~6~~

~~$6^2 - 1$~~

~~$6^3 - 2\cdot 6$ or $5\cdot 6^2 + 4\cdot 6$ or $6(6^2-2)$ or $6(5\cdot 6+4)$~~

~~$6^4 - 3\cdot 6^2 + 1$ or $5\cdot 6^3 + 3\cdot 6^2 + 1$~~

~~$6^5 - 4\cdot 6^3 + 3\cdot 6$ or $5\cdot 6^4 + 2\cdot 6^3 + 3\cdot 6$~~

~~$6^6 - 5\cdot 6^4 + 6\cdot 6^2 - 1$ or $5\cdot 6^5 + 6^4 + 6^3 + 1$~~

~~$6^7 - 6\cdot 6^5 + 6^4 + 4 6^3 - 15$ or $6^7 - 6^6 + 2\cdot 6^4 - 2\cdot 6^3 + 1$~~

~~or $6^7 - 5\cdot 6^5 - 4\cdot 6^4 - 7\cdot 6^2 - 4$~~

$F_1 = 6$

$F_2 = 6^2 - 1$ ✓

$F_3 = 6^3 - 2\cdot 6$ ✓ $= \quad 6(6^2-2)$

$F_4 = 6^4 - 3\cdot 6^2 + 1$ ✓

$F_5 = 6^5 - 4\cdot 6^3 + 3\cdot 6$ ✓ $= 6(6^4 - 4\cdot 6^2 + 3)$

$F_6 = 6^6 - 5 6^4 + 6\cdot 6^2 - 1$ ✓ $= \; 6^6 - 5 6^4 + 6^3 - 1$

$F_7 = 6^7 - 6\cdot 6^5 + 10\cdot 6^3 - 4\cdot 6$ ✓ $= 6^7 - 6^6 + 6^4 + 4 6^3 - 4\cdot 6 = 6(6^6 - 6^5 + 6^4 + 4 6^2 - 4)$

$F_8 = 6^8 - 7\cdot 6^6 + 15\cdot 6^4 - 10\cdot 6^2 + 1$ ✓ $= 6^8 - 6^7 - 6^6 + 2\cdot 6^5 + 3\cdot 6^4 - 6^3 - 4\cdot 6^2 + 1$

$F_9 = 6^9 - 8\cdot 6^7 + \overset{15}{21}\cdot 6^5 - 20\cdot 6^3 + 5\cdot 6$ ✓ $= 6(6^8 - 6^7 - 2\cdot 6^6 + 3\cdot 6^5 + 3\cdot 6^4 - 3\cdot 6^3 - 2\cdot 6^2 + 5)$

$F_{10} = 6^{10} - 9 6^8 + 28\cdot 6^6 - 35\cdot 6^4 + 15\cdot 6^2 - 1$ ✓ $= 6^{10} - 6^9 - 3\cdot 6^8 + 4\cdot 6^7 + 4\cdot 6^6 - 5\cdot 6^5 - 5\cdot 6^4$
$+ 2\cdot 6^3 + 3\cdot 6^2 - 1$

EXPLICIT
DIRECT FORMULA

~~B(n)~~ $=$ $B(n) =$

$$F_N = 6^N - \frac{(N-1)!}{(N-2)!\,1!}\,6^{N-2} + \frac{(N-2)!}{(N-4)!\,2!}\,6^{N-4} - \frac{(N-3)!}{(N-6)!\,3!}\,6^{N-6} + \frac{(N-4)!}{(N-8)!\,4!}\,6^{N-8} - \cdots$$

$$- \frac{(N-5)!}{(N-10)!\,5!}\,6^{N-10} + \cdots \qquad 6^{n-n}$$

$p = 5.826427$

1.0303300858

$\shortparallel$

Negative powers $= 0$
Negative factorial $= 1$ $\quad 0! = 1$

---

also $B(n) = p^n\left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^6} + \cdots\right) = p^n\,\frac{p^2}{p^2-1}$ integral part

reconcile

when $p = 3 + \sqrt{8}$

$B(n+1) = p\,B(n)$, $\quad B(n+2) = p^2\,B(n)$, $\quad B(n+r) = p^r B(n)$

$B(n) = 6\;B(n-1) - B(n-2)$

$B(n+2) = 6\;B(n+1) - B(n)$

$P+K = B-1$

$B+K = R$

$L=1$ Fulcrum Numbers

L = 1 Recursion Formulae

$B_n + K_n = R_n$

1) ~~$B_n = R_n - K_n$~~

$\Delta B = \Delta R - \Delta K$ $\quad \Delta^2 B = \Delta^2 R - \Delta^2 K$

$R_n + B_n = K_{n+1}$

$R_n - B_n = K_n$

where $\Delta B = B_n - B_{n-1}$

$\Delta^2 B = \Delta B_n - \Delta B_{n-1}$

2) $2B_n = K_{n+1} - K_n$

$2R_n = K_{n+1} + K_n$

3) $B_{n+1} = 7B_n + R_{n-1} - R_n$

$8B_n = R_{n+1} - R_{n-1}$

$B_n - B_{n-1} = 2K_n + 1$

4) ~~$B_{n+3} = (B_{n+1} + B_{n+2} - 6)/B_n$~~

5) ~~$B_{2n-1} = B_n(B_n - B_{n-2}) - 1$~~

$B_{n+1} + B_n$ + [illegible] $= R_{n+1} - R_n$

6) ~~$B_{2n-1} = B_n^2 - B_{n-1}^2$~~

$B_n + B_{n-1} = R_n - R_{n-1}$

7) ~~$B_{2n} = B_n(B_{n+1} - B_{n-1})$~~

$B_n - B_{n-1} = 2K_n + 1$

$2B_n = R_n - R_{n-1} + 2K_n + 1$

8) $R_{n+1} = 8B_n + R_{n-1}$

9) $R_{n+3} = 10(2K_{n+2} + 1) - R_n - 1$

10) $R_{n+3} = 10(B_{n+2} - B_{n+1}) - R_n - 1$

11) $R_{2n-1} = (R_n - R_{n-1})^2$

12) $R_{2n} = B_n(R_{n+1} - R_{n-1})$

$2R_n = K_n + K_{n+1}$

13) $R_{2n} = 8B_n^2$

14) $8R_{2n} = (R_{n+1} - R_{n-1})^2$

B_n

$2B_n$

$8B_n$

B_{2n-1}

B_{2n}

R_n

15) $B_{n+1}B_{n+2} - B_nB_{n+3} = B_2 = 6$

16) $R_{n+1}B_n - R_nB_{n+1} = R_n + B_n = K_{n+1}$

17) $2(1 + B_2 + B_3 + \ldots\ldots + B_{n-1}) = R_n - B_n = K_n$

18) $B_{n+1} - B_n = 2(R_{n+1} - B_{n+1}) + 1 = 2K_{n+1} + 1$

$\sqrt{R_{2n-1}} = R_n - R_{n-1} = 7B_n - B_{n+1}$

$(R_n - R_{n-1})(R_{n+1} - R_n) = R_{2n}$

$B_n + K_n = R_n$

$P_n + K_n = B_n - 1$

$P_n + R_n = 2B_n - 1$

✓

FULCRUM2.WPD → FULNVM02 JULY 21, 2003

In the special case where **L** = 1, equation **2)** becomes

$J = B - 1$

$$\sum_{1}^{B-1} n = \sum_{1}^{B+K} n - \sum_{1}^{B} n$$

$R = B + K$

General Recursion Formulae

Hence, $\mathbf{B(B-1) = (B+K)(B+K+1) - B(B+1)}$

$B_n = 6B_{n-1} - B_{n-2}$

or $\mathbf{B^2 = (2K+1)B + K^2 + K}$

$K_n = 6K_{n-1} - K_{n-2} + 2$

$R_N = 6R_{N-1} - R_{N-2} + 2$

$P_N = 6P_{N-1} - P_{N-2} + 2$

The following quantities are the first ten integ~~ral~~er solutions of this equation:
[R = B + K], [P = B - 1 - K]

B	# above B $K_0 = 0$	H H+1 FIRST $R = K + B$	$P + K = B - 1$ ~~P = # < B~~
$B_1 = 1$	$K_1 = 0$	$R_1 = 1$	$P_1 = 0$
$B_2 = 6$	$K_2 = 2$	$R_2 = 8$	$P_2 = 3$
$B_3 = 35$	$K_3 = 14$	$R_3 = 49$	$P_3 = 20$
$B_4 = 204$	$K_4 = 84$	$R_4 = 288$	$P_4 = 119$
$B_5 = 1189$	$K_5 = 492$	$R_5 = 1681$	$P_5 = 696$
$B_6 = 6930$	$K_6 = 2870$	$R_6 = 9800$	$P_6 = 4059$
$B_7 = 40391$	$K_7 = 16730$	$R_7 = 57121$	$P_7 = 23660$
$B_8 = 235416$	$K_8 = 97512$	$R_8 = 332928$	$P_8 = 137903$
$B_9 = 1372105$	$K_9 = 568344$	$R_9 = 1940449$	$P_9 = 803760$
$B_{10} = 7997214$	$K_{10} = 3312554$	$R_{10} = 11309768$	$P_{10} = 4684659$

$H = B + K$

For example, the sum of the integers less tha~~t~~n the fulcrum number $B_3 = 35$ equals (35 x 34)/2 = 595;. and the sum of the integers above 35 to 35 + 14 = 49 is (49 x 50)/2 - (35 x 36)/2, which is equal to 1225 - 630 = 595. etc.

$n = 14$
$d = 34$

Limit of ratio of integers = rational

Here rational minus irrational → 0

Ratios

As n increases without limit, the ratio $R_n/B_n \longrightarrow \sqrt{2} = 1.414213562373$

As n increases without limit, the ratio $B_n/K_n \longrightarrow (1+\sqrt{2}) = 2.414213562373$

As n increases without limit, the ratio $R_n/K_n \longrightarrow (2+\sqrt{2}) = 3.414213562373$

As n increases without limit, the ratio $P_n/K_n \dashrightarrow \sqrt{2} = 1.414213562373$

As n increases without limit, the ratio $B_n/P_n \longrightarrow (1+\sqrt{2})\sqrt{2} = 1.707106781186$

As n increases without limit, the ratio $B_n/B_{n-1} \longrightarrow (3+2\sqrt{2}) = 5.828427124746$

As n increases without limit, the ratio $R_n/R_{n-1} \longrightarrow (3+2\sqrt{2}) = 5.828427124746$

As n increases without limit, the ratio $K_n/K_{n-1} \longrightarrow (3+2\sqrt{2}) = 5.828427124746$

As n increases without limit, the ratio $P_n/P_{n-1} \longrightarrow (3+2\sqrt{2}) = 5.828427124746$

Limit of $\frac{B_n}{B_{n-1}}$

a{rationals}

$= 3+2\sqrt{2}$

Note that in all the above ratios, as n increases without limit, the ratios of integer quantities approach irrational values involving $\sqrt{2}$..

limit of rational = irrational

$\lim_{n \to \infty} \frac{B_n}{B_{n-1}} = 3 + \sqrt{8}$

$B^2 - (2K+1)B - (K^2+K) = 0$

~~B = K or K+1~~

What ratio → $5+\sqrt{15}$ etc ?
Fulcrum?

Iterated ratios

As n increases, we have,
From I.

$$B_n/B_{n-1} \longrightarrow 3 + \sqrt{8}\text{, hence } B_n/B_{n-1} \times B_{n-1}/B_{n-2} = B_n/B_{n-2} \longrightarrow (3+\sqrt{8})^2 = 17 + 12\sqrt{2}$$

Similarly, From II.

$$R_n/R_{n-1} \longrightarrow 3 + \sqrt{8}\text{, and } R_n/R_{n-2} \longrightarrow 17 + 12\sqrt{2} = 33.970562748477$$

$$4 + 3\sqrt{2} = (3 + 2\sqrt{2})\sqrt{2} = 8.242640687119$$
$$17 + 12\sqrt{2} = 33.970562748477$$
$$24 + 17\sqrt{2} = (17 + 12\sqrt{2})\sqrt{2} = 48.041630560342$$

Note the inversions:

$$(3 + 2\sqrt{2})^{-1} = 3 - 2\sqrt{2}$$
$$(4 + 3\sqrt{2})^{-1} = (3\sqrt{2} - 4)/2$$
$$(17 + 12\sqrt{2})^{-1} = 17 - 12\sqrt{2}$$
$$(24 + 17\sqrt{2})^{-1} = (17\sqrt{2} - 24)/2$$

$$R_{2m} = (2 \times B_m)^2 \times 2$$
$$= 8 B_m^2$$
$$R_{2m+1} = (B_m + B_{m+1})^2$$

Factoring the above quantities B and R into their prime factors, we get:

		H		
$B_1 = 1$	1	$R_1 = 1$	1	
$B_2 = 6$	2 x 3	$R_2 = 8$	2^3	$3^2 - 1 = 2 \times 2^2$
$B_3 = 35$	5 x 7	$R_3 = 49$	7^2	
$B_4 = 204$	2^2 x 3 x 17	$R_4 = 288$	$2^5 \times 3^2$	$17^2 - 1 = 2 \times 12^2$
$B_5 = 1189$	29 x 41	$R_5 = 1681$	41^2	
$B_6 = 6930$	2 x 3^2 x 5 x 7 x 11	$R_6 = 9800$	$2^3 \times 5^2 \times 7^2$	$99^2 - 1 = 2 \times 70^2$
$B_7 = 40391$	13^2 x 239	$R_7 = 57121$	239^2	
$B_8 = 235416$	2^3 x 3 x 17 x 577	$R_8 = 332928$	$2^7 \times 3^2 \times 17^2$	$577^2 - 1 = 2 \times 408^2$
$B_9 = 1372105$	5 x 7 x 197 x 199	$R_9 = 1940449$	$7^2 \times 199^2 = 1393^2$	
$B_{10} = 7997214$	2 x 3 x 19 x 29 x 41 x 59	$R_{10} = 11309768$	$2^3 \times 29^2 \times 41^2$	$3363^2 - 1$ $= (2378)^2 \times 2$

Note that the odd numbered R's are all perfect squares. are fulcrum numbers And evens are perfect squares -1
Factoring the quantities K and P into their prime factors we get,

$K_1 = 0$	0	$P_1 = 0$	0
$K_2 = 2$	2	$P_2 = 3$	3
$K_3 = 14$	2 x 7	$P_3 = 20$	2^2 x 5
$K_4 = 84$	2^2 x 3 x7	$P_4 = 119$	7 x 17
$K_5 = 492$	2^2 x 3 x 41	$P_5 = 696$	2^3 x 3 x 29
$K_6 = 2870$	2 x 5 x 7 x 41	$P_6 = 4059$	3^2 x 11 x 41
$K_7 = 16730$	2 x 5 x 7 x 239	$P_7 = 23660$	2^2 x 5 x 7 x 13^2
$K_8 = 97512$	2^3 x 17 x 717	$P_8 = 137903$	239 x 577
$K_9 = 568344$	2^3 x 3 x 7 x 17 x 199	$P_9 = 803760$	2^4 x 3 x 5 x 17 x 197
$K_{10} = 3312554$	2 x 7 x 29 x 41 x 199	$P_{10} = 4684659$	3 x 7 x 19 x 59 x 199

$$B_n = \frac{p^n - q^n}{2\sqrt{8}}$$

FULCRUM4.WPD → FULNUMO2 ✓ where $p = 3+\sqrt{8}$, $q = 3-\sqrt{8}$ August 10, 2003

~~A direct~~ An EXPLICIT formula for fulcrum numbers of the class **L = 1:** (Ramanujan Numbers)

$$\mathbf{1)}\quad \mathbf{B(n)} = \sum_{s=0}^{\infty} \mathbf{p}^{n-2s} = \mathbf{p}^n\left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \frac{1}{p^6} + \ldots\right)$$

where $\mathbf{p = 3 + \sqrt{8}}$ = 5.8284271247461900976033774484194

The series converges and can be summed,

$$\textbf{Let}\qquad \mathbf{S} = \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \frac{1}{p^6} + \ldots\right)$$

This is roots of previous page

then approximate EXPLICIT FORMULA

ROUND B(n) for exact

$$\mathbf{S} = \frac{p^2}{p^2 - 1}\ \checkmark \qquad \textbf{and} \qquad \boxed{\mathbf{B(n) = p^n S}}$$

$$S = p\bar{S}$$

where $\mathbf{S = (17 + 12\sqrt{2})/(16 + 12\sqrt{2})}$ = 1.0303300858899106433006332715786 4

Recursive formulae for fulcrum numbers are then,

$$\mathbf{B(n+1) = p\,B(n)},\quad \mathbf{B(n+2) = p^2\,B(n)},\quad \textbf{and}\quad \mathbf{B(n+r) = p^r\,B(n)}$$

Round $[p^n S]$
to give $B(n)$
when $p = 3+\sqrt{8}$
$S = \dfrac{p^2}{p^2-1}$

and also without p

$$B(n) = 6\,B(n-1) - B(n-2)$$

$$B(n+1) = 6\,B(n) - B(n-1) = p\,B(n)$$

$$(6 - p)B(n) = B(n-1)$$

work this backwards

$$B(n+1) = \frac{B(n)}{6-p} = p\,B(n)$$

$$B(n+1) = \frac{B(n)}{3-\sqrt{8}}$$

page 4

$$6p - p^2 = 1$$

$$p^2 - 6p + 1 = 0$$

$$p = \frac{6 \pm \sqrt{36-4}}{2} = 3+\sqrt{8}\ \checkmark$$

$$= \sqrt{32} = 2\sqrt{8}$$

denominator

$$\frac{B(n+1)}{B(n)} = 3+\sqrt{8} = \frac{1}{3-\sqrt{8}}$$

$p = 3 + \sqrt{8}$ $\qquad$ $q = 3 - \sqrt{8}$

$$S = \frac{p^2}{p^2 - 1}$$

$$A = \sqrt{} = 2\sqrt{8} = \sqrt{32} = \sqrt{j^2 + 4k}$$

$$A p = 6\sqrt{8} + 16$$
$$p^2 - 1 = 16 + 6\sqrt{8}$$

$$A \cdot p = p^2 - 1 \qquad \frac{1}{A} = \frac{p}{p^2 - 1} \qquad \frac{p}{A} = S$$

EXPLICIT

$$B(n) = \overset{trunc}{p^{n-1} S} \qquad B(n) = \frac{\overset{trunc}{p^{n}}}{A} \qquad \text{EXACT} \quad \frac{p^n - q^n}{A}$$

$$A = p - q$$

RAMANUJAN

RECURSION FORMULAE

HOW ARE ①, ②, ... to be derived from the primary?

PRIMARY: $B_n = 6B_{n-1} - B_{n-2}$

If $\begin{cases} B_n = 0,\ 1,\ 6,\ 35,\ 204,\ 1189,\ 6930,\ 40391,\ 235,416, \\ n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9, \end{cases}$

① $B_n(B_{n+1} - B_{n-1}) = B_{2n-1}$ $\quad n \geq 1$ OK

② $B_n^2 - B_{n-1}^2 = B_{2n-2}$ $\quad n \geq 2$ OK

③ $B_{n+1} \cdot B_{n+2} - B_n \cdot B_{n+3} = 6$ $\quad n \geq 1$ OK

④ $5(B_n + B_{n+1}) = B_{n-1} + B_{n+2}$ $\quad n \geq 2$ OK

⑤ $B_n(B_n + B_{n-2}) = B_{2n-2} + 1$ $\quad n \geq 3$ OK

from ② and ⑤

⑥ $B_{n-1}^2 = B_n B_{n-2} + 1$ $\quad n \geq 3$ OK

~~from ① and ③~~

~~⑦ $B_{n-1}(B_{n+2} - B_n) = B_{2n-1} - 6$ $\quad n \geq 2$~~

General $x^2 + bx + c = 0$
$u + v = -b$
$u \cdot v = c$

RAMANUJAN ITERATION FORMULAE

$x^2 - 6x + 1 = 0$

$A_2 = \sqrt{6A_1 - 1}$ $\qquad$ $A_2 = \frac{A_1^2 + 1}{6}$

RAMANUJAN

$v = 3 + \sqrt{8} = 5.828427$

$u = 3 - \sqrt{8} = 0.171573$

$u + v = 6$

$u \cdot v = 1$

FIBONACCI

$v = \frac{1+\sqrt{5}}{2}$

$u = \frac{1-\sqrt{5}}{2}$

$u + v = 1$

$u \cdot v = -1$

$\Pi\Sigma$ $\quad u + v = L$ $\quad$ ProdSum
$u \cdot v = L$

$\Pi\Delta$ $\quad u + v = L$
$u \cdot v = -L$

RAMANUJAN NUMBERS
and associates

$B_n - D_n = B_{n-1}$

$\Sigma_1 B_n = Z_{2n}$

$\Delta_1 B_n = W_{2n}$

$D_n = W_{2n-2} \quad n>1$

$\Sigma_2 B_n = 8 B_{n+1}$

$\Delta_2 B_n = 4 B_{n+1}$

$B_{n+1} - B_n = W_{2n}$

$W_n \cdot Z_n = B_n$

$Z_n + W_n = W_{n+1} \qquad n>1$

$Z_n - W_n = W_{n-1}$

$2Z_n = W_{n+1} + W_{n-1}, \quad 2W_n = W_{n+1} - W_{n-1}$

$\Sigma_1 W_n = Z_{n+1}$

$\Delta_1 W_n = Z_n$

$\Sigma_1 Z_n = 2W_{n+1}$

$\Delta_1 Z_n = 2W_n$

$\Sigma_2 W_n = 2W_{n+2}$

$\Delta_2 W_n = 2W_n$

$\Sigma_2 Z_n = 2Z_{n+2}$

$\Delta_2 Z_n = 2Z_n$

$[0,1] \quad B_{n+2} = 6B_{n+1} - B_n$

$[1,1] \quad D_{n+2} = 6D_{n+1} - D_n$

$[0,1] \quad W_{n+2} = 2W_{n+1} + W_n$

$[1,1] \quad Z_{n+2} = 2Z_{n+1} + Z_n$

~~$Z_{n+1} W_n + W_{n+1} Z_n = W_{2n+2}$~~

$W_{n+1} Z_n - Z_{n+1} W_n = 1$

$W_{n+1} Z_n + Z_{n+1} W_n = W_{2n}$

$2 W_{n+1} Z_n = W_{2n} + 1$

$$\begin{aligned} B_{n+2} = Z_{n+2} \cdot W_{n+2} &= (2Z_{n+1} + Z_n)(2W_{n+1} + W_n) \\ &= 4Z_{n+1} W_{n+1} + Z_n W_n + 2(Z_{n+1} W_n + W_{n+1} Z_n) \\ &= 4Z_{n+1} W_{n+1} + W_n Z_n + 2W_{2n} \\ &= 4B_{n+1} + B_n + 2(B_{n+1} - B_n) \end{aligned}$$

$\therefore \; B_{n+2} = 6B_{n+1} - B_n$

Definitions for this page

$B \sim \begin{bmatrix} 6 & -1 \\ 0 & 1 \end{bmatrix} \qquad D \sim \begin{bmatrix} 6 & -1 \\ 1 & 1 \end{bmatrix}$

$W \sim \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad Z \sim \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

The following are some recursion formulae for L = 1 fulcrum numbers:

B formulae

① **1L)** $B_n(B_{n+1} - B_{n-1}) = B_{2n-1}$ ✓ $n \geq 1$

⑤ **2L)** $B_n(B_n - B_{n-2}) = B_{2n-2} + 1$ ✓ $n \geq 3$

② **3L)** $B_n^2 - B_{n-1}^2 = B_{2n-2}$ ✓ $n \geq 2$

③ **4L)** $B_{n+1} B_{n+2} - B_n B_{n+3} = B_2 = 6$ ✓ $n \geq 1$

THE GENERAL RECURSION FORMULA

$$\boxed{B_N = 6 B_{N-1} - B_{N-2}}$$

R formulae

5L) $(R_n - R_{n-1})^2 = R_{2n-1}$

B and R formulae

6L) $R_{n+1} B_n - R_n B_{n+1} = R_n + B_n$

7L) $B_n (R_{n+1} - R_{n-1}) = R_{2n}$

8L) $2(1 + B_2 + B_3 + \ldots.. + B_{n-1}) = R_n - B_n$

9L) $8 B_n = R_{n+1} - R_{n-1}$

10L) $8 B_n^2 = R_{2n}$

11L) $10(2K_{n+2} + 1) = R_n + R_{n+3} + 1$

12L) $10(B_{n+2} - B_{n+1}) = R_n + R_{n+3} + 1$

13L) $B_{n+1} - B_n = 2(R_{n+1} - B_{n+1}) + 1 = 2K_{n+1} + 1$

Prediction formulae

14L $B_{n+1} = 7B_n + R_{n-1} - R_n$

15L) $R_{n+1} = 8B_n + R_{n-1}$

16L) $K_{n+1} = B_n + R_n = 2B_n + K_n$

17L) $R_{n+3} = 10(B_{n+2} - B_{n+1}) - R_n - 1$

18L) $B_{n+3} = (B_{n+1} + B_{n+2} - 6)/B_n$

L = 1 Recursion Formulae

$B_n = 6B_{n-1} - B_{n-2}$

1) $B_n = R_n - K_n$

2) $2B_n = K_{n+1} - K_n$

3) $B_{n+1} = 7B_n + R_{n-1} - R_n$

4) $B_{n+3} = (B_{n+1} * B_{n+2} - 6)/B_n$ ③

5) $B_{2n-2} = B_n(B_n - B_{n-2}) - 1$ ✓ ⑤

6) $B_{2n-2} = B_n^2 - B_{n-1}^2$ ✓ ②

7) $B_{2n-1} = B_n(B_{n+1} - B_{n-1})$ ✓ ①

foretell the future

8) $R_{n+1} = 8B_n + R_{n-1}$

9) $R_{n+3} = 10(2K_{n+2} + 1) - R_n - 1$

10) $R_{n+3} = 10(B_{n+2} - B_{n+1}) - R_n - 1$

11) $R_{2n-1} = (R_n - R_{n-1})^2$

12) $R_{2n} = B_n (R_{n+1} - R_{n-1})$

13) $R_{2n} = 8 B_n^2$

14) $8R_{2n} = (R_{n+1} - R_{n-1})^2$ Future

15) $B_{n+1} B_{n+2} - B_n B_{n+3} = B_2 = 6$

16) $R_{n+1} B_n - R_n B_{n+1} = R_n + B_n = K_{n+1}$

17) $2 (1 + B_2 + B_3 + ...+ B_{n-1}) = R_n - B_n = K_n$ $\sum_{i=1}^{n-1} B_i = \frac{K_n}{2}$

18) $B_{n+1} - B_n = 2(R_{n+1} - B_{n+1}) + 1 = 2K_{n+1} + 1$

✓

RAMFORM.WPD

January 1, 2004

RAMANUJAN NUMBERS

RECURSION FORMULAE

$B_1 = 0,\ B_2 = 1,\ B_3 = 6,\ B_4 = 35, \ldots$ How are recursion formulae derived from 1) ?

1) $B_n = 6\,B_{n-1} - B_{n-2}$ ✓

① 2) $B_{2n-1} = B_n(B_{n+1} - B_{n-1})$ ✓

② 3) $B_{2n-2} = B_n^{\,2} - B_{n-1}^{\,2}$ ✓

⑤ 4) $B_{2n-2} = B_n^{\,2} - B_nB_{n-2} - 1$ ✓

③ 5) $B_{n+1}B_{n+2} - B_nB_{n+3} = 6$ ✓

$n \geq 3$ 6) $B_{n-1}^{\,2} = B_nB_{n-2} + 1$ from 3) and 4) ~~Or $B_n = (B_{n-1}^{\,2} - 1) / B_{n-2}$~~

? 7) $B_{2n} = B_{n-1}(B_{n+2} - B_n) + 6$? from 2) and 5) ✗

8) $B_n = 3\,B_{n-1} + \checkmark(8\,B_{n-1}^{\,2} + 1)$ from 1) and 6)
8) infers that $(8\,B_n^{\,2} + 1)$ is a square for all B_n

EXPLICIT FORMULAE

$B_n = \mathrm{trunc}\left(\dfrac{p^{n+1}}{p^2-1}\right)$

$B_n = \mathrm{trunc}\,(p^{n-1}\,S)$ where trunc(x) is the integer part of x
where $p = 3 + \checkmark 8$
where $S = p^2 / (p^2 - 1) = (17 + 12\checkmark 2) / (16 + 12\checkmark 2)$

giving the additional recursion formulae:

$B_{n+1} = \mathrm{trunc}\,(p\,B_n)$ and $B_{n+m} = \mathrm{trunc}\,(p^m\,B_n)$

$B_1 = 1$
$B_2 = 6$
$B_3 = 35$
$B_4 = 204$
$B_5 = 1189$
$B_6 = 6930$
$B_7 = 40391$
$B_8 = 235416$
$B_9 = 1372105$
$B_{10} = 7997214$

Note last digit sequence 1,6,5,4,9,0,1,6,5,4,.....

January 4, 2004

RAMANUJAN NUMBERS
B(n)
Explicit Formulae

$p := 3 + \sqrt{8}$ $q := 3 - \sqrt{8}$

p^{2n} etc ?
cf Fibonaccis

APPROX EXPLICIT

$$A(n) := \frac{p^n}{2\sqrt{8}}$$

EXACT EXPLICIT

$$B(n) := \frac{(p^n - q^n)}{2\sqrt{8}}$$

A second approx explicit on next page

$n := 1, 2 .. 15$

n =	A(n) =	round(A(n)) =	B(n) =
1	1.030330086	1	1
2	6.00520382	6	6
3	35.000892834	35	35
4	204.000153186	204	204
5	1189.000026283	1189	1189
6	6930.000004509	6930	6930
7	40391.000000774	40391	40391
8	235416.000000133	235416	235416
9	1372105.00000002	1372105	1372105
10	7997214	7997214	7997214
11	$4.6611179 \cdot 10^7$	46611179	46611179
12	$2.7166986 \cdot 10^8$	271669860	271669860
13	$1.583407981 \cdot 10^9$	$1.583407981 \cdot 10^9$	$1.583407981 \cdot 10^9$
14	$9.228778026 \cdot 10^9$	$9.228778026 \cdot 10^9$	$9.228778026 \cdot 10^9$
15	$5.378926017 \cdot 10^{10}$	$5.378926018 \cdot 10^{10}$	$5.378926017 \cdot 10^{10}$

$$\bar{S} := \frac{1}{2\sqrt{8}} = \frac{p}{p^2-1} = \frac{S}{p}$$

where $S = \frac{p^2}{p^2-1}$

$p = 5.828427125$ ✓

$q = 0.171572875$ ✓

$p + q = 6$

$\bar{S} = 0.176776695 = \frac{1}{\sqrt{32}}$ ✓ [illegible] $= \frac{1}{2\sqrt{8}}$

$$\cancel{\bar{S} := \frac{p^2}{p^2 - 1}}$$ ~~error~~

$$\cancel{\bar{S} := \frac{(3+\sqrt{8})^2}{16 + 6\sqrt{8}}}$$

change designation do not use S

B(n) Recursion Formula

$B(n) := 6 \cdot B(n-1) - B(n-2)$

This $\bar{S}$ and the S on the next page are not the same

We seem to have two explicit formula the one on this page and on the next page

$A(n) = \frac{p^n}{2\sqrt{8}}$ gives same values as $B(n) = p^{n-1} S$

FULNUM03.WPD ✓ August 19, 2003

i.e. $\frac{p}{p^2-1} = \frac{1}{2\sqrt{8}}$ $= .1767767$

Comparison of values:

approximate
EXPLICIT FORMULA
$B(n) = p^{n-1}S$

approximate
ON
RECURSI~~VE~~ FORMULA
$B(n) = pB(n-1)$

$\mathbf{p = 3 + \sqrt{8}}$ = 5.8284271247461900976033774484194

$\mathbf{S = p^2 / (p^2 - 1)}$ = 1.03033008588991064330063327157864

B(1) = 1	B(1) = 1
B(2) = 6.00520382004282	B(2) = 5.82842712474619
B(3) = 35.00089283436705	B(3) = 34.97056274847714
B(4) = 204.00015318615948	B(4) = 203.99494936611665
B(5) = 1189.00002628258983	B(5) = 1188.99913344822277
B(6) = 6930.00000450937950	B(6) = 6929.99985132322002
B(7) = 40391.00000077368720	B(7) = 40390.99997449109737
B(8) = 235416.00000013274373	B(8) = 235415.99999562336423
B(9) = 1372105.00000002277522	B(9) = 1372104.99999924908801
B(10 = 7997214.00000000390761	B(10 = 7997213.99999987116387
B(11)= 46611179.00000000067044	B(11) = 46611178.99999997789521
B(12)= 271669860.00000000011502	B(12) = 271669859.99999999620741
B(13)= 1583407981.00000000001973	B(13) = 1583407980.99999999934929
B(14)= 9228778026.00000000000338	B(14) = 9228778025.99999999988835
B(15)= 53789260175.00000000000058	B(15) = 53789260174.99999999998084
B(16)= 313506783024.00000000000009	B(16) = 313506783023.99999999999671

B(17)= 1827251437969.0000000000001709907
B(18)= 10650001844790.000000000000029154

Numbers in the right hand column are = corresponding numbers in the left hand column divided by S

notation: B_n = round (B(n))
where round = replace with nearest integer

Exact recursion formula: $B_n = 6 B_{n-1} - B_{n-2}$

What is the exact explicit formula? $B_n = \frac{p^n - q^n}{2\sqrt{8}}$ ✓ where $q = 3-\sqrt{8}$

~~$= \sqrt{a_2}$~~

~~$r_n = a_1 \pm \sqrt{a_2}$~~

Values for fulcrum numbers:

Recursion formulae: $\mathbf{R_{n+1} = 8B_n + R_{n-1}}$

$\mathbf{K_{n+1} = B_n + R_n}$

$\mathbf{B_{n+1} = R_{n+1} - K_{n+1}}$

The following quantities are the first twenty integral solutions of the equation:

$$\mathbf{B^2 = (2K + 1)B + K(K + 1)}$$

$B_1 = 1$	$K_1 = 0$	$R_1 = 1$
$B_2 = 6$	$K_2 = 2$	$R_2 = 8$
$B_3 = 35$	$K_3 = 14$	$R_3 = 49$
$B_4 = 204$	$K_4 = 84$	$R_4 = 288$
$B_5 = 1189$	$K_5 = 492$	$R_5 = 1681$
$B_6 = 6930$	$K_6 = 2870$	$R_6 = 9800$
$B_7 = 40391$	$K_7 = 16730$	$R_7 = 57121$
$B_8 = 235416$	$K_8 = 97512$	$R_8 = 332928$
$B_9 = 1372105$	$K_9 = 568344$	$R_9 = 1940449$
$B_{10} = 7997214$	$K_{10} = 3312554$	$R_{10} = 11309768$
$B_{11} = 46611179$	$K_{11} = 19306982$	$R_{11} = 65918161$
$B_{12} = 271669860$	$K_{12} = 112529340$	$R_{12} = 384199200$
$B_{13} = 1583407981$	$K_{13} = 655869060$	$R_{13} = 2239277041$
$B_{14} = 9228778026$	$K_{14} = 3822685022$	$R_{14} = 13051463048$
$B_{15} = 53789260175$	$K_{15} = 22280241074$	$R_{15} = 76069501249$
$B_{16} = 313506783024$	$K_{16} = 129858761424$	$R_{16} = 443365544448$
$B_{17} = 1827251437969$	$K_{17} = 756872327472$	$R_{17} = 2584123765441$
$B_{18} = 10650001844790$	$K_{18} = 4411375203410$	$R_{18} = 15061377048200$
$B_{19} = 62072759630771$	$K_{19} = 25711378892990$	$R_{19} = 87784138523761$
$B_{20} = 361786555939836$	$K_{20} = 149856898154532$	$R_{20} = 511643454094368$

$3 + 2\sqrt{2} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 377\ 448\ 419$

$B_{12}/B_{11} = 5.828\ 427\ 124\ 746\ 190\ 179$
$B_{14}/B_{13} = 5.828\ 427\ 124\ 746\ 190\ 097\ 673$
$B_{16}/B_{15} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 438$
$B_{18}/B_{17} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 377\ 501$
$B_{20}/B_{19} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 377\ 448\ 465$

$R_{12}/R_{11} = 5.828\ 427\ 161\ 370\ 597\ 095$
$R_{16}/R_{15} = 5.828\ 427\ 124\ 777\ 927\ 042$
$R_{20}/R_{19} = 5.828\ 427\ 124\ 746\ 217\ 599$

$K_{12}/K_{11} = 5.828\ 427\ 249\ 789\ 739\ 276$
$K_{16}/K_{15} = 5.828\ 427\ 124\ 854\ 546\ 804$
$K_{20}/K_{19} = 5.828\ 427\ 124\ 746\ 283\ 994$

FULNUM03.WPD

Fulcrum #	FULOGS.WPD LOG$_{10}$
B_1 = 1	0
B_2 = 6	0.778151250383643632508766797979608
B_3 = 35	1.54406804435027563549847736386814
B_4 = 204	2.30963016742589875626267558703244
B_5 = 1189	3.07518185461869158184225861293744
B_6 = 6930	3.84073323461180674605247203634589
B_7 = 40391	4.60628460556181122341467147603255
B_8 = 235416	5.3718359762456113799652481493208
B_9 = 1372105	6.1373873469215752129702434660051
B_{10} = 7997214	6.90293871759730836614801527120912
B_{11} = 46611179	7.66849008827303472874569185012236
B_{12} = 271669860	8.43404145894876089144735417973836
B_{13} = 1583407981	9.19959282962448704826462725948071
B_{14} = 9228778026	9.96514420030021320490868009281612
B_{15} = 53789260175	10.7306955709759393615476337981892
B_{16} = 313506783024	11.4962469416516655181864373992499
B_{17} = 1827251437969	12.2617983123273916748252365816522
B_{18} = 10650001844790	13.0273496830031178314640356339813
B_{19} = 62072759630771	13.7929010536788439881028346824815
B_{20} = 361786555939836	14.5584524243545701447416337308689
B_{21} = 2108646576008245	15.324003795030296301380432779253
B_{22} = 12290092900109634	16.089555165706022458019231827637
B_{23} = 71631910824649559	16.855106536381748614658030876021
B_{24} = 417501372047787720	17.6206579070574747712968299244051
B_{25} = 2433376321462076761	18.3862092777332009279356289727891
B_{26} = 14182756556724672846	19.1517606484089270845744280211731
B_{27} = 82663163018885960315	19.9173120190846532412132270695571
B_{28} = 481796221556591089044	20.6828633897603793978520261179411
B_{29} = 2808114166320660573949	21.4484147604361055544908251663251
B_{30} = 16366888776367372354650	22.2139661311118317111296242147091
B_{31} = 95393218491883573553951	22.9795175017875578677684232630931
B_{32} = 555992422174934068969056	23.7450688724632840244072223114772
B_{33} = 3240561314557720840260385	24.5106202431390101810460213598612
B_{34} = 18887375465171390972593254	25.2761716138147363376848204082452
B_{35} = 110083691476470624995299139	26.0417229844904624943236194566292
B_{36} = 641614773393652358999201580	26.8072743551661886509624185050132
B_{37} = 3739604948885443528999910341	27.5728257258419148076012175533972
B_{38} = 21796014919919008815000260466	28.3383770965176409642400166017812
B_{39} = 127036484570628609361001652455	29.1039284671933671208788156501652
B_{40} = 740422892503852647351009654264	29.8694798378690932775176146985493
B_{41} = 4315500870452487274745056273129	30.6350312085448194341564137469333
B_{42} = 25152582330211071001119327984510	31.4005825792205455907952127953173

Fulcrum Numbers

1) $B(n) = S\,p^{n-1}$

$p^2 = 17 + 6\sqrt{8} =$ 33.9705627484771405856202646905164

$p = 3 + \sqrt{8} =$ 5.82842712474619009760337744841885

$S = p^2/(p^2 - 1) = (17 + 12\sqrt{2})/(16 + 12\sqrt{2}) =$ 1.03033008588991064330063327157864

B(2) = 6.00520382004282697870358853894046

B(3) = 35.0008928343670512289208979620668

B(4) = 204.000153186159480394821799233352

B(5) = 1189.00002628258983114000989743778

B(6) = 6930.0000045093795064452375853917

B(7) = 40391.0000007736872075314156149031

- - - - - - - - - - - - -

B(13) = 1583407981.00000000001973591163252

B(21) = 2108646576008244.99999999999998861

B(31) = 95393218491883573553950.9999992555

B(36) = 641614773393652358999201579.994224

B(38) = 21796014919919008815000260465.7933 This and all above by 1)

- - - - - - - - - - - - -

Explicit Formula

Trunc $B(n) = Sp^{n-1}$

or Round

$\frac{p^n - q^n}{2\sqrt{8}}$

Same $p = 3 + \sqrt{8}$

FULCRUM3.WPD → FULNUM03 August 4, 2003

Values for fulcrum numbers:

Recursion formulae: $\mathbf{R_{n+1} = 8B_n + R_{n-1}}$

$\mathbf{K_{n+1} = B_n + R_n}$

$\mathbf{B_{n+1} = R_{n+1} - K_{n+1}}$

$K_{n+1} - K_n = 2B_n$

$K_n = \frac{(B_n - 1) - B_{n-1}}{2}$

The following quantities are the first twenty integral solutions of the equation:

$$\mathbf{B^2 = (2K + 1)B + K(K + 1)}$$

$K = \frac{1}{2}\left[\sqrt{8B^2+1} - (2B+1)\right]$

$B_1 = 1$	$K_1 = 0$	$R_1 = 1$	
$B_2 = 6$	$K_2 = 2$	$R_2 = 8$	$3^2-1 = 2^2 \cdot 2$ $2 \cdot 3$
$B_3 = 35$	$K_3 = 14$	$R_3 = 49$	7^2 $49+1 = 50$ $\frac{50}{2} = 5^2$ $7 \cdot 5$
$B_4 = 204$	$K_4 = 84$	$R_4 = 288$	$17^2-1 = 12^2 \cdot 2$ $17 \cdot 12$
$B_5 = 1189$	$K_5 = 492$	$R_5 = 1681$	$= 41^2$ $\frac{1682}{2} = 841 = 29^2$ $29 \cdot 41$
$B_6 = 6930$	$K_6 = 2870$	$R_6 = 9800$	$99^2-1 = (9 \times 11)^2-1$
$B_7 = 40391$	$K_7 = 16730$	$R_7 = 57121$	$= 239^2$
$B_8 = 235416$	$K_8 = 97512$	$R_8 = 332928$	$= 577^2-1$
$B_9 = 1372105$	$K_9 = 568344$	$R_9 = 1940449$	$= 1393^2 = (7 \times 199)^2$
$B_{10} = 7997214$	$K_{10} = 3312554$	$R_{10} = 11309768$	
$B_{11} = 46611179$	$K_{11} = 19306982$	$R_{11} = 65918161$	
$B_{12} = 271669860$	$K_{12} = 112529340$	$R_{12} = 384199200$	odds x^2
$B_{13} = 1583407981$	$K_{13} = 655869060$	$R_{13} = 2239277041$	Evens x^2-1
$B_{14} = 9228778026$	$K_{14} = 3822685022$	$R_{14} = 13051463048$	
$B_{15} = 53789260175$	$K_{15} = 22280241074$	$R_{15} = 76069501249$	
$B_{16} = 313506783024$	$K_{16} = 129858761424$	$R_{16} = 443365544448$	
$B_{17} = 1827251437969$	$K_{17} = 756872327472$	$R_{17} = 2584123765441$	
$B_{18} = 10650001844790$	$K_{18} = 4411375203410$	$R_{18} = 15061377048200$	
$B_{19} = 62072759630771$	$K_{19} = 25711378892990$	$R_{19} = 87784138523761$	
$B_{20} = 361786555939836$	$K_{20} = 149856898154532$	$R_{20} = 511643454094368$	

$3 + 2\sqrt{2} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 377\ 448\ 419$

Odd R's are perfect squares

∴ are fulcrum numbers

$B_{12}/B_{11} = 5.828\ 427\ 124\ 746\ 190\ 179$

$B_{14}/B_{13} = 5.828\ 427\ 124\ 746\ 190\ 097\ 673$

$B_{16}/B_{15} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 438$

$B_{18}/B_{17} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 377\ 501$

$B_{20}/B_{19} = 5.828\ 427\ 124\ 746\ 190\ 097\ 603\ 377\ 448\ 465$

For Perfect Squares

$B = (K+1)^2$

R. K

e.g. $B = 49$ $K = 6$

6 # up to 55 = 315

7 # down to 42 = 315

$R_{12}/R_{11} = 5.828\ 427\ 161\ 370\ 597\ 095$

$R_{16}/R_{15} = 5.828\ 427\ 124\ 777\ 927\ 042$

$R_{20}/R_{19} = 5.828\ 427\ 124\ 746\ 217\ 599$

$K_{12}/K_{11} = 5.828\ 427\ 249\ 789\ 739\ 276$

$K_{16}/K_{15} = 5.828\ 427\ 124\ 854\ 546\ 804$

$K_{20}/K_{19} = 5.828\ 427\ 124\ 746\ 283\ 994$

B

$d=7$ $\boxed{49} = 7^2$

$n=6$

$\sum = 315$ $\sum = 315$

$\frac{48 \cdot 49}{2} = 1176$ $\frac{55 \cdot 56}{2} = 1540$

$\frac{41 \cdot 42}{2} = 861$ $\frac{49 \cdot 50}{2} = 1225$

315 315

B

$\sum = 68060$

$d=41$ $\boxed{1681} = 41^2$ $n = 40$ $\sum = 68060$

$\frac{1660 \cdot 1681}{2} = 1412040$ $\frac{1721 \times 1722}{2} = 1481781$

$\frac{1639 \cdot 1640}{2} = 1343980$ $\frac{1681 \times 1682}{2} = 1413721$

68060 68060

For perfect squares s

$\therefore$ $d = \sqrt{s}$, $n = \sqrt{s} - 1$

2

$n := 1, 2 .. 20$

fulcr2.mcd 03-08-10

$p := 3 + \sqrt{8}$

precision setting 12

$p = 5.8284271247$

values are correct up to n = 13

$$B(n) := \left(p^{n} + p^{n-2} + p^{n-4} + p^{n-6} + p^{n-8} + p^{n-10} + p^{n-12}\right)$$

n =	round(B(n), 0) =
1	6
2	35
3	204
4	$1.189\cdot10^{3}$
5	$6.93\cdot10^{3}$
6	$4.0391\cdot10^{4}$
7	$2.35416\cdot10^{5}$
8	$1.372105\cdot10^{6}$
9	$7.997214\cdot10^{6}$
10	$4.6611179\cdot10^{7}$
11	$2.7166986\cdot10^{8}$
12	$1.583407981\cdot10^{9}$
13	$9.228778026\cdot10^{9}$
14	$5.3789260174\cdot10^{10}$
15	$3.13506783018\cdot10^{11}$
16	$1.827251437934\cdot10^{12}$
17	$1.065000184459\cdot10^{13}$
18	$6.207275962958\cdot10^{13}$
19	$3.617865559329\cdot10^{14}$
20	$2.108646575968\cdot10^{15}$

FULNUM03

also Ramanujan

Fulcrum #	~~FULOGS.WPD~~	LOG$_{10}$
$B_{1} = 1$		0
$B_{2} = 6$		0.778151250383643632508766797979608
$B_{3} = 35$		1.54406804435027563549847736386814
$B_{4} = 204$		2.30963016742589875626267558703244
$B_{5} = 1189$		3.07518185461869158184225861293744
$B_{6} = 6930$		3.84073323461180674605247203634589
$B_{7} = 40391$		4.60628460556181122341467147603255
$B_{8} = 235416$		5.3718359762456113799652481493208
$B_{9} = 1372105$		6.1373873469215752129702434660051
$B_{10} = 7997214$		6.90293871759730836614801527120912
$B_{11} = 46611179$		7.66849008827303472874569185012236
$B_{12} = 271669860$		8.43404145894876089144735417973836
$B_{13} = 1583407981$		9.19959282962448704826462725948071
$B_{14} = 9228778026$		9.96514420030021320490868009281612
$B_{15} = 53789260175$		10.7306955709759393615476337981892
$B_{16} = 313506783024$		11.4962469416516655181864373992499
$B_{17} = 1827251437969$		12.2617983123273916748252365816522
$B_{18} = 10650001844790$		13.0273496830031178314640356339813
$B_{19} = 62072759630771$		13.7929010536788439881028346824815
$B_{20} = 361786555939836$		14.5584524243545701447416337308689
$B_{21} = 2108646576008245$		15.324003795030296301380432779253
$B_{22} = 12290092900109634$		16.089555165706022458019231827637
$B_{23} = 71631910824649559$		16.855106536381748614658030876021
$B_{24} = 417501372047787720$		17.6206579070574747712968299244051
$B_{25} = 2433376321462076761$		18.3862092777332009279356289727891
$B_{26} = 14182756556724672846$		19.1517606484089270845744280211731
$B_{27} = 82663163018885960315$		19.9173120190846532412132270695571
$B_{28} = 481796221556591089044$		20.6828633897603793978520261179411
$B_{29} = 2808114166320660573949$		21.4484147604361055544908251663251
$B_{30} = 16366888776367372354650$		22.2139661311118317111296242147091
$B_{31} = 95393218491883573553951$		22.9795175017875578677684232630931
$B_{32} = 555992422174934068969056$		23.7450688724632840244072223114772
$B_{33} = 3240561314557720840260385$		24.5106202431390101810460213598612
$B_{34} = 18887375465171390972593254$		25.2761716138147363376848204082452
$B_{35} = 110083691476470624995299139$		26.0417229844904624943236194566292
$B_{36} = 641614773393652358999201580$		26.8072743551661886509624185050132
$B_{37} = 3739604948885443528999910341$		27.5728257258419148076012175533972
$B_{38} = 21796014919919008815000260466$		28.3383770965176409642400166017812
$B_{39} = 127036484570628609361001652455$		29.1039284671933671208788156501652
$B_{40} = 740422892503852647351009654264$		29.8694798378690932775176146985493
$B_{41} = 4315500870452487274745056273129$		30.6350312085448194341564137469333
$B_{42} = 25152582330211071001119327984510$		31.4005825792205455907952127953173
$B_{43} = 146599993110813938731970911633931$		31.1661339498962717474340118437013
$B_{44} = 854447376334672561390706141819076$		32.9316853205719979040728108920853
$B_{45} = 4980084264897221429612265939280520$		33.6972366912477240607116099404693

NOTE THE LAST FIGURE SEQUENCE

1, 6, 5, 4, 9, 0, 1, 6, 5, 4, 9, 0, ...

FULNUM03

Calc1.wpd August 11, 2003

$J = B - 1$

Fulcrum Numbers

1) $B(n) = S\,p^n$ 2) $B(n+2) = p^2\,B(n)$ 3) $B(n+1) = p\,B(n)$

$p^2 = 17 + 6\sqrt{8} =$ 33.9705627484771405856202646905164

$p = 3 + \sqrt{8} =$ 5.82842712474619009760337744841885

$S = p^2/(p^2 - 1) = (17 + 12\sqrt{2})/(16 + 12\sqrt{2}) =$ 1.03033008588991064330063327157864

B(1) = 6.00520382004282697870358853894046

B(2) = 35.0008928343670512289208979620668

B(3) = 204.000153186159480394821799233352

B(4) = 1189.0002628258983114000989743778

B(5) = 6930.000045093795064452375853917

B(6) = 40391.0000007736872075314156149031

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B(12) = 1583407981.00000000001973591163252

B(20) = 2108646576008244.99999999999998861

B(30) = 95393218491883573553950.9999992555

B(35) = 641614773393652358999201579.994224

B(37) = 21796014919919008815000260465.7933 This and all above by 1)

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B(37) = 21796014919919008815000260465.7989 by the recursion formula 2)

B(39) = 740422892503852647351009654256.625 by 1)

B(39) = 740422892503852647351009654256.727 by 2)

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